

Chapter 7

Propagation of Waves

7.1 Wavefronts

7.1.1 Plane Waves

The form of any wave (matter or electromagnetic) is determined by its source and described by the shape of its wavefront, *i.e.*, the locus of points of constant phase. If a traveling wave is emitted by a planar source, then the points of constant phase form a plane surface parallel to the face of the source. Such a wave is called a *plane wave*, and travels in one direction (ideally). Since energy is conserved, the total energy in the wave must equal the energy emitted by the source, and therefore the *energy density* (the energy passing through a unit area), is constant for a plane wave. Recall that in a wave of amplitude A and frequency ω , the energy $\mathcal{E} \propto A^2\omega^2$. Therefore, for a plane wave, the amplitude is constant; the wave does not attenuate.

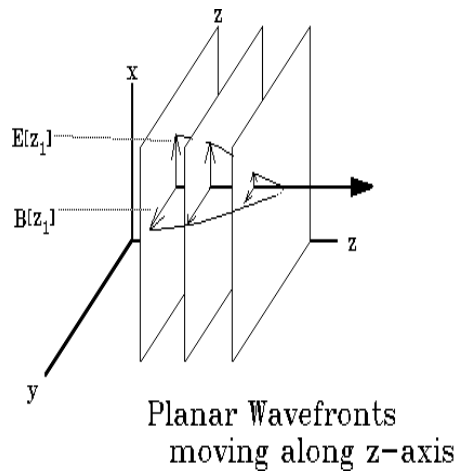
Plane wave toward $z = +\infty$ at velocity $v_\phi = \frac{\omega}{k}$, wavelength $\lambda = \frac{2\pi}{k}$, frequency $\nu = \frac{\omega}{2\pi}$, amplitude A_0 :

$$f[x, y, z, t] = A_0 \cos [kz \mp \omega t]$$

(n.b., no variation in y or z)

General 3-D plane wave traveling in a direction $\mathbf{k} = [k_x, k_y, k_z]$, $\mathbf{r} = [x, y, z]$

$$f[\mathbf{r}, t] = A_0 \cos [\mathbf{k} \cdot \mathbf{r} - \omega t] \implies \mathbf{k} \cdot \mathbf{r} = k_x x + k_y y + k_z z$$



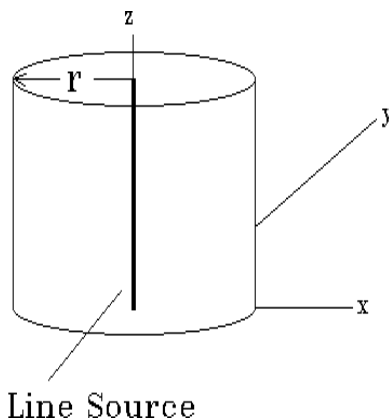
7.1.2 Cylindrical Waves

If a wave is emitted from a line source, the wavefronts are *cylindrical*. Since the wave expands to fill a cylinder of radius r_0 , the wavefront crosses a cylindrical area that grows as $Area = 2\pi r h \propto r$. Therefore, since energy is conserved, the energy per unit area must decrease as r increases:

$$\begin{aligned} \frac{\mathcal{E}}{Area} &= constant = \frac{\mathcal{E}}{2\pi r h} \propto \frac{\mathcal{E}}{r} \propto \frac{A_0^2}{r} = constant \\ \Rightarrow amplitude &\propto \frac{A_0}{\sqrt{r}} \end{aligned}$$

The equation for a cylindrical wavefront emerging from (or collapsing into) a line source is:

$$\begin{aligned} f[x, y, z, t] &= A[r] \cos [kr \mp \omega t] \\ &= \frac{A_0}{\sqrt{r}} \cos [kr \mp \omega t] \\ r &= \sqrt{x^2 + y^2} > 0 \\ \text{"-"} &\Rightarrow \text{emerging} \\ \text{"+"} &\Rightarrow \text{collapsing} \\ A_0 &= \text{amplitude at } r = 0 \end{aligned}$$



Cylindrical waves expanding from a line source.

7.1.3 Spherical Waves

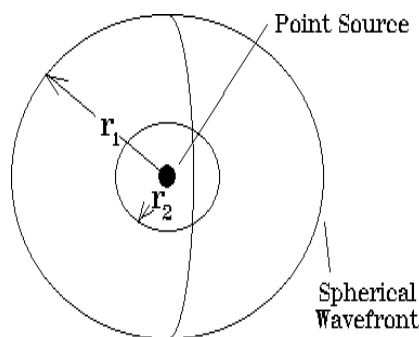
The wavefront emerging from (or collapsing into) a point is spherical. The area the wave must cross increases as $x^2 + y^2 + z^2 = r^2$ (area of sphere is $4\pi r^2$). Therefore the energy density drops as r^2 and

the amplitude of the wave must decrease as $\frac{1}{r}$. The equation for a spherical wave is

$$\begin{aligned} f[x, y, z, t] &= f[\mathbf{r}, t] = A[r] \cos[kr \mp \omega t] \\ &= \frac{A_0}{r} \cos[kr \mp \omega t], \text{ where } r > 0 \\ \text{"-"} &\implies \textit{emerging} \\ \text{"+"} &\implies \textit{collapsing} \\ A_0 &= \textit{amplitude at } r = 0 \end{aligned}$$

Note the pattern for the amplitude of plane, cylindrical, and spherical waves:

$$\begin{aligned} \text{plane wave} &\implies \text{2-D source (plane)} \implies \text{amplitude } A[r] \propto r^{-0} = 1 \\ \text{cylindrical wave} &\implies \text{1-D source (line)} \implies A[r] \propto r^{-\frac{1}{2}} \\ \text{spherical wave} &\implies \text{0-D source (point)} \implies A[r] \propto r^{-1} \end{aligned}$$



Spherical waves expanding from a point source.

7.2 Huygens' Principle

I, §1, §3

The spherical wave is the basic wave for light propagation using Huygens' principle. In 1678, Christiaan Huygens theorized a model for light propagation that claimed that each point on a propagating wavefront (regardless of "shape") could be assumed to be a source of a new spherical wave. The sum of these secondary spherical "wavelets" produced the subsequent wavefronts. Huygens' principle had the glaring disadvantage that these secondary spherical wavefronts propagated "backwards" as well as forwards. This problem was later solved by Fresnel and Kirchhoff in the 19th century. With that correction, the Huygens' model provides a very useful model for light propagation that naturally leads to expressions for "diffracted" light.

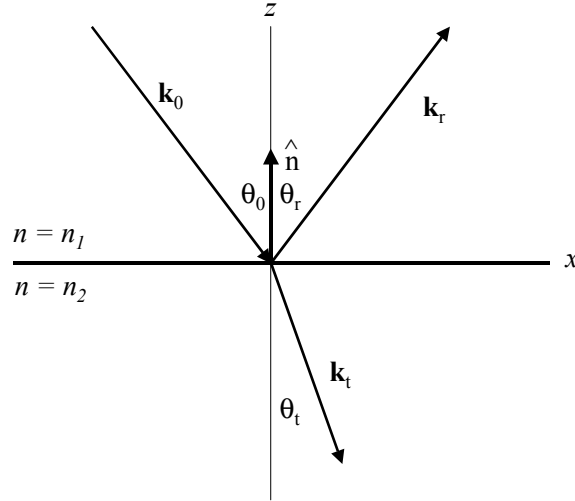
7.3 Electromagnetic Waves at a Interface between Media

We can easily observe that a light wave incident on an interface between two media "creates" two other waves: the refracted and reflected wave. The equations that determine the equations for these three electromagnetic waves are not difficult to derive, though the process is perhaps tedious. The equations determine the properties of light on either side of the interface, including:

1. Equal angles of incidence and reflection;
2. Snell's Law that relates the incident and refracted wave;
3. Relative intensities of the three waves;

4. States of polarization of the three waves; and
5. Phases of the three light waves.

The mathematics are based on Maxwell's equations for the three waves and the continuity conditions that must be satisfied at the boundary. For simplicity, we consider only plane waves, so that each light wave is described by a single wavevector \mathbf{k} that points in the direction of propagation. The interface is located at $z = 0$ and the incident wavevector \mathbf{k}_0 , the reflected vector \mathbf{k}_r , the transmitted vector \mathbf{k}_t and the unit vector $\hat{\mathbf{n}}$ normal to the interface are shown:



The \mathbf{k} vectors of the incident, reflected, and “transmitted” (refracted) wave at the interface between two media of index n_1 and n_2 .

As drawn, the normal to the surface is the vector with components:

$$\hat{\mathbf{n}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The incident electric field is:

$$\begin{aligned} \mathbf{E}_{\text{incident}} &= \mathbf{E}_0 e^{i(\mathbf{k}_0 \cdot \mathbf{r} - \omega t)} \\ \mathbf{E}_{\text{reflected}} &= \mathbf{E}_r e^{i(\mathbf{k}_r \cdot \mathbf{r} - \omega t + \phi_r)} \\ \mathbf{E}_{\text{transmitted}} &= \mathbf{E}_t e^{i(\mathbf{k}_t \cdot \mathbf{r} - \omega t + \phi_t)} \end{aligned}$$

where $\mathbf{r} = [x, y, z]$ is the position vector and ϕ_r and ϕ_t are the (perhaps different) initial phases of the light waves. The magnitudes of the wave vectors in the two media differ because the wavelength is scaled:

$$\begin{aligned} |\mathbf{k}_0| &= \frac{\omega}{v_1} = \frac{2\pi}{\lambda_1} = \frac{2\pi n_1}{\lambda_0} \\ |\mathbf{k}_r| &= \frac{\omega}{v_1} = \frac{2\pi}{\lambda_1} = \frac{2\pi n_1}{\lambda_0} = |\mathbf{k}_0| \\ |\mathbf{k}_t| &= \frac{\omega}{v_2} = \frac{2\pi}{\lambda_2} = \frac{2\pi n_2}{\lambda_0} \end{aligned}$$

where λ_0 is the wavelength in vacuum.

The waves must all have the same phase at the interface located at $z = 0$:

$$(\mathbf{k}_0 \cdot \mathbf{r} - \omega t)|_{z=0} = (\mathbf{k}_r \cdot \mathbf{r} - \omega t + \phi_r)|_{z=0} = (\mathbf{k}_t \cdot \mathbf{r} - \omega t + \phi_t)|_{z=0}$$

which immediately implies that the temporal frequencies of the three waves must be identical – the “color” of the light does not change as the light travels into a different medium. Therefore the spatial vectors must satisfy the conditions:

$$(\mathbf{k}_0 \bullet \mathbf{r})|_{z=0} = (\mathbf{k}_r \bullet \mathbf{r} + \phi_r)|_{z=0} = (\mathbf{k}_t \bullet \mathbf{r} + \phi_t)|_{z=0}$$

Since the scalar product of the three wavevectors with the same vector must be equal, then the three vectors must lie in the same plane.

The difference of a pair of terms must vanish, which requires that:

$$\begin{aligned} ((\mathbf{k}_0 - \mathbf{k}_r) \bullet \mathbf{r})|_{z=0} &= \phi_r \\ ((\mathbf{k}_0 - \mathbf{k}_t) \bullet \mathbf{r})|_{z=0} &= \phi_t \end{aligned}$$

Since the scalar products of the differences of the wavevectors and the position vector \mathbf{r} are constants, this implies that the position vectors that satisfy these conditions lie in a plane perpendicular to the differences of the wavevectors. The pairs of wavevectors must lie in the same plane perpendicular to the interface, which ensures that:

$$\begin{aligned} \hat{\mathbf{n}} \times (\mathbf{k}_0 - \mathbf{k}_r) &= \hat{\mathbf{n}} \times \mathbf{k}_0 - \hat{\mathbf{n}} \times \mathbf{k}_r = 0 \\ &= |\hat{\mathbf{n}}| |\mathbf{k}_0| \sin[\theta_0] - |\hat{\mathbf{n}}| |\mathbf{k}_r| \sin[\theta_r] \\ &= |\mathbf{k}_0| \sin[\theta_0] - |\mathbf{k}_r| \sin[\theta_r] \\ &= \frac{2\pi n_1}{\lambda_0} \sin[\theta_0] - \frac{2\pi n_1}{\lambda_0} \sin[\theta_r] \\ \implies \sin[\theta_0] &= \sin[\theta_r] \\ \implies \theta_0 &= \theta_r \end{aligned}$$

By following the same argument for the second example, we obtain Snell’s law for refraction:

$$\begin{aligned} \hat{\mathbf{n}} \times (\mathbf{k}_0 - \mathbf{k}_t) &= \hat{\mathbf{n}} \times \mathbf{k}_0 - \hat{\mathbf{n}} \times \mathbf{k}_t = 0 \\ &= |\hat{\mathbf{n}}| |\mathbf{k}_0| \sin[\theta_0] - |\hat{\mathbf{n}}| |\mathbf{k}_t| \sin[\theta_t] \\ &= \frac{2\pi n_1}{\lambda_0} \sin[\theta_0] - \frac{2\pi n_2}{\lambda_0} \sin[\theta_t] \\ \implies n_1 \sin[\theta_0] &= n_2 \sin[\theta_t] \end{aligned}$$

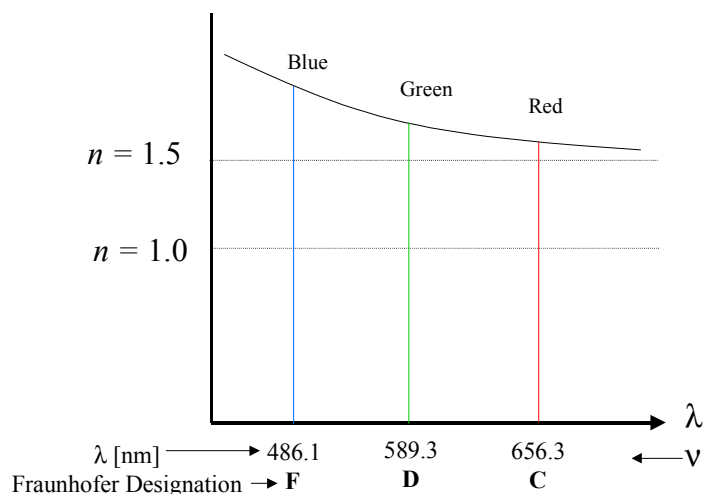
7.4 Refraction

We have already stated that the index of refraction n relates the phase velocity of light in vacuum with that in matter:

$$n = \frac{c}{v} \geq 1.$$

In a dispersive medium, the index n decreases with increasing λ , which ensures that the *phase velocity* $\frac{\omega}{k}$ (of the average wave) is larger than the *group velocity* $\frac{d\omega}{dk}$ (of the modulation wave).

Refraction is the result of the interaction of light with atoms in the medium and depends on wavelength because the refractive index is also; recall that the index decreases with increasing wavelength:



Typical dispersion curve for glass showing the decrease in n with increasing λ and the three spectral wavelengths used to specify “refractivity”, “mean dispersion”, and “partial dispersion”.

To a first approximation, the index of refraction varies as λ^{-1} , which allows us to write an empirical expression for the *refractivity* of the medium $n - 1$:

$$n[\lambda] - 1 \simeq a + \frac{b}{\lambda}$$

where a and b are parameters determined from measurements. The observation that the index increases with λ demonstrates that the parameter $b > 0$. Cauchy came up with an empirical relation for the refractivity more free parameters:

$$n - 1 \simeq A \left(1 + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + \dots \right)$$

Again, the behavior of normal dispersion ensures that A and B are both positive. Yet a better formula was proposed by Hartmann:

$$n \simeq n_0 + \frac{\alpha}{(\lambda - \lambda_0)^{1.2}}$$

The refractive properties of the glass are approximately specified by the refractivity and the measured differences in refractive index at the three Fraunhofer wavelengths F , D , and C :

<i>Refractivity</i>	$n_D - 1$	$1.75 \leq n_D \leq 1.5$
<i>Mean Dispersion</i>	$n_F - n_C > 0$	differences between blue and red indices
<i>Partial Dispersion</i>	$n_D - n_C > 0$	differences between yellow and red indices
<i>Abbé Number</i>	$\nu \equiv \frac{n_D - 1}{n_F - n_C}$	ratio of refractivity and mean dispersion, $25 \leq \nu \leq 65$

Glasses are specified by six-digit numbers $abcdef$, where $n_D = 1.abc$, to three decimal places, and $\nu = de.f$. Note that larger values of the refractivity mean that the refractive index is larger and thus so is the deviation angle in Snell’s law. A larger Abbé number means that the mean dispersion is smaller and thus there will be a smaller difference in the angles of refraction. Such glasses with larger Abbé numbers and smaller indices and less dispersion are *crown* glasses, while glasses with smaller Abbé numbers are *flint* glasses, which are “denser”. Examples of glass specifications include Borosilicate crown glass (BSC), which has a specification number of 517645, so its refractive index in the D line is 1.517 and its Abbé number is 64.5. The specification number for a common flint glass is 619364, so $n_D = 1.619$ (relatively large) and $\nu = 36.4$ (smallish). Now consider the refractive indices in the three lines for two different glasses.

<i>Line</i>	λ [nm]	n for Crown	n for Flint
C	656.28	1.51418	1.69427
D	589.59	1.51666	1.70100
F	486.13	1.52225	1.71748

For the crown glass :

$$\text{refractivity } n_D - 1 = 0.51666$$

$$\nu = \frac{1.51666 - 1}{1.52225 - 1.51418} \simeq 64.022$$

$$\text{Glass number} = 516640$$

For the flint glass :

$$\text{refractivity } n_D - 1 = 0.70100$$

$$\nu = \frac{0.70100 - 1}{1.71748 - 1.69427} \simeq 30.202$$

$$\text{Glass number} = 701302$$

7.4.1 Optical Path Length

Because the phase velocity of light in a medium is less than that in vacuum, light takes longer to travel through a given thickness of material than through the same “thickness” of vacuum. For a fixed distance d , we know that:

$$\begin{aligned} d &= v \cdot t \text{ (distance=velocity} \cdot \text{time)} \\ &= ct_1 \text{ (in vacuum)} \\ &= \frac{c}{n}t_2 \text{ (in medium of index } n) \\ \implies t_1 &= \frac{t_2}{n} \implies t_2 > t_1 \end{aligned}$$

In the time t_2 required for light to travel the distance d in a material of index n , light would travel a longer distance $nd = ct_2$ in vacuum. The distance nd traveled in vacuum in the equivalent time is the *optical path length* in the medium.

7.4.2 Reflection and Transmission Coefficients

As already discussed, radiation is both reflected and refracted at an interface. The relative “amounts” reflected and refracted are determined by the reflection and transmission coefficients ρ and t , which are derived by solving Maxwell’s equations at the interface to ensure continuity of the electric and magnetic fields. The *amplitude reflection coefficient* is defined as the ratio of the amplitude of the reflected electric field to the incident field:

$$E_R = \rho E_0 \implies \rho = \frac{E_R}{E_0}$$

If the wave is *normally incident* on the interface, the reflection coefficient is shown to be:

$$\rho = \frac{n_1 - n_2}{n_1 + n_2}$$

Example:

If the input medium has a smaller refractive index n (a *rarer* medium) than the second (*denser*) medium, then the amplitude reflection coefficient ρ is:

$$\begin{aligned}
 n_1 &= 1.0 \\
 n_2 &= 1.5 \\
 \implies \rho &= \frac{1.0 - 1.5}{1.0 + 1.5} = -0.2 = 0.2e^{i\pi} \\
 &\text{for "rare-to-dense" reflection}
 \end{aligned}$$

In words, the phase of the reflected light is changed by π radians = 180° if reflected at a “rare-to-dense” interface.

If the input medium is “denser” (larger index n), then the amplitude reflection coefficient is positive:

$$\begin{aligned}
 n_1 &= 1.5 \\
 n_2 &= 1.0 \\
 \implies \rho &= \frac{1.5 - 1.0}{1.5 + 1.0} = +0.2
 \end{aligned}$$

There is no phase shift of the reflected amplitude in “dense-to-rare” reflection, commonly called “internal” reflection..

The *intensity* of the reflected light is the squared-magnitude of the amplitude,

$$I_r = |E_r|^2 = |\rho E_0|^2 = \rho^2 |E_0|^2$$

and is therefore proportional to ρ^2 . The intensity of the reflection for normal incidence onto a glass-air interface includes no phase shift, i.e., ($n_1 = 1$, $n_2 = 1.5$ or $n_1 = 1.5$ and $n_2 = 1$),

$$I_r = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2 I_0 = (-0.2)^2 I_0 = 0.04 I_0.$$

Therefore, 4% of the intensity is reflected at a glass-air interface. The behavior of ρ and the analogous “transmission coefficient” τ for different incident angles are considered in the next section.

7.5 POLARIZATION

Maxwell’s equations demonstrated that light is a *transverse* wave (as opposed to longitudinal waves, *e.g.*, sound). Both the \mathbf{E} and \mathbf{B} vectors are perpendicular to the direction of propagation of the radiation. Even before Maxwell, Thomas Young inferred the transverse character of light in 1817 when he passed light through a calcite crystal (calcium carbonate, $CaCO_3$). Two beams emerged from the crystal, which Young brilliantly deduced were orthogonally polarized, *i.e.*, the directions of the \mathbf{E} vectors of the two beams are orthogonal. The two components of an electromagnetic wave are the electric field \mathbf{E} [$\frac{\text{volts}}{\text{meter}}$] and the magnetic field \mathbf{B} [$\text{tesla} = \frac{\text{webers}}{\text{m}^2}$].

The *polarization* of radiation is defined as the plane of vibration of the electric vector \mathbf{E} , rather than of \mathbf{B} , because the effect of the \mathbf{E} field on a free charge (*i.e.*, electron) is much greater than the effect of \mathbf{B} . This is seen from the Lorentz equation, or the Lorentz force law:

$$\begin{aligned}
 \mathbf{F} &\propto q_0 \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \\
 q_0 &= \text{charge [coulombs]} \\
 \mathbf{F} &= \text{force on the charge [newtons} = \frac{\text{kg}\cdot\text{m}}{\text{s}^2}] \\
 \mathbf{v} &= \text{velocity of the charge } q_0 \left[\frac{\text{m}}{\text{s}} \right] \\
 c &= \text{velocity of light } [3 \cdot 10^8 \frac{\text{m}}{\text{s}}]
 \end{aligned}$$

The factor c^{-1} ensures that the force on the electron due to the magnetic field is usually much smaller than the electric force.

7.5.1 PLANE POLARIZATION = LINEAR POLARIZATION

The most familiar type of polarization is linear polarization, where the \mathbf{E} -vector oscillates in the same plane at all points on the wave.

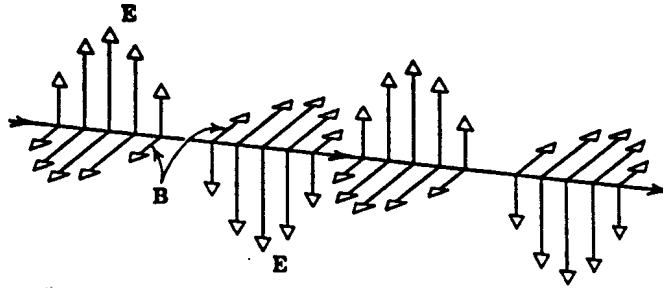
Any state of linear polarization can be expressed as a linear combination (sum) of two orthogonal states (*basis states*), *e.g.*, the x - and y -components of the \mathbf{E} -vector for a wave traveling toward $z = \pm\infty$:

$$\begin{aligned}\mathbf{E} &= \mathbf{E}[\mathbf{r}, t] = [\hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y] \cos[kz - \omega t] \\ \hat{\mathbf{x}}, \hat{\mathbf{y}} &= \text{unit vectors along } x \text{ and } y \\ E_x, E_y &= \text{amplitudes of the } x\text{- and } y\text{-components of } \mathbf{E}.\end{aligned}$$

For a wave of amplitude E_0 polarized at an angle θ relative to the x -axis:

$$\begin{aligned}E_x &= E_0 \cos[\theta] \\ E_y &= E_0 \sin[\theta]\end{aligned}$$

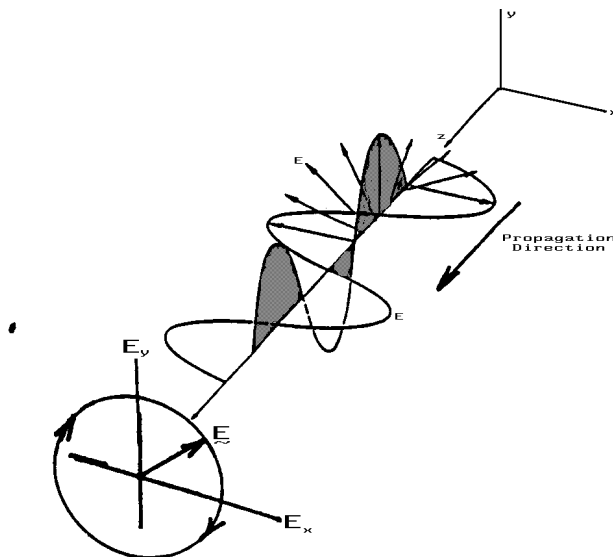
Linearly polarized radiation oscillates in the same plane at all times and at all points in space. Especially note that E_x and E_y are *in phase* for linearly polarized light, *i.e.*, both components have zero-crossings at the same point in time and space.



Electric field vector \mathbf{E} and magnetic field vector \mathbf{H} of a plane-polarized wave

7.5.2 CIRCULAR POLARIZATION

If the \mathbf{E} -vector describes a helical (*i.e.*, screw-like) motion in space, the projection of the \mathbf{E} -vector onto a plane normal to the propagation direction exhibits circular motion over time, hence circular polarization.



Circular polarization occurs when the electric fields along orthogonal axes have the same amplitude by their phases differ by $\pm\frac{\pi}{2}$ radians.

If we sit at a fixed point in space $z = z_0$, the motion of the \mathbf{E} -vector is the sum of two orthogonal linearly polarized states, but with one component out-of-phase by $90^\circ = \frac{\pi}{2}$ radians. The math is identical to that used to describe oscillator motion as the projection of rotary motion:

$$\text{motion} = \hat{\mathbf{x}} \cos[\omega t] + \hat{\mathbf{y}} \cos\left[\omega t \mp \frac{\pi}{2}\right] = \hat{\mathbf{x}} \cos[\omega t] \pm \hat{\mathbf{y}} \sin[\omega t]$$

For a traveling wave:

$$\begin{aligned} \mathbf{E} &= [E_x, E_y] = \left[E_0 \cos[kz - \omega t], E_0 \cos\left[kz - \omega t \mp \frac{\pi}{2}\right] \right] \\ &= [E_0 \cos[kz - \omega t], \pm E_0 \sin[kz - \omega t]] \end{aligned}$$

where the upper sign applies to right-handed circular polarization (angular momentum convention)

7.5.3 NOMENCLATURE FOR CIRCULAR POLARIZATION

Like linearly polarized light, circularly polarized light has two orthogonal states, *i.e.*, clockwise and counterclockwise rotation of the \mathbf{E} -vector. These are termed *right-handed* (RHCP) and *left-handed* (LHCP). There are two conventions for the nomenclature:

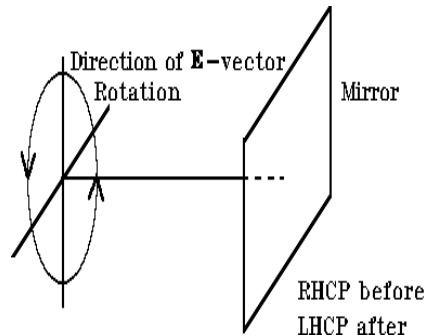
1. Angular Momentum Convention (my preference): Point the thumb of the $\left\{ \begin{array}{l} \text{right} \\ \text{left} \end{array} \right\}$ hand in the direction of propagation. If the fingers point in the direction of rotation of the \mathbf{E} -vector, then the light is $\left\{ \begin{array}{l} \text{RHCP} \\ \text{LHCP} \end{array} \right\}$.
2. Optics (screwy) Convention: The path traveled by the \mathbf{E} -vector of RHCP light is the same path described by a right-hand screw. Of course, the natural laws defined by Murphy ensure that the two conventions are opposite: RHCP light by the angular momentum convention is LHCP by the screw convention.

7.5.4 ELLIPTICAL POLARIZATION, REFLECTIONS

If the amplitudes of the x - and y -components of the E -vector are not equal, or if the phase difference is not $\pm\frac{\pi}{2} = \pm 90^\circ$, then the projection of the path of the \mathbf{E} -vector is not a circle, but rather an ellipse. This results in *elliptical polarization*. Note that elliptical polarization may be either right- or left-handed, as defined above.

7.5.5 Change of Handedness on Reflection

By conservation of angular momentum, the direction of rotation of the \mathbf{E} -vector does not change on reflection. Since the direction of propagation reverses, the handedness of the circular or elliptical polarization changes:



Change in “handedness” of a circularly polarized wave upon reflection by a mirror.

Natural Light

The superposition of emissions from a large number of thermal source elements (as in a light bulb) has a random orientation of polarizations. The state of polarization of the resulting light varies randomly over a very short time scale ($\approx 10^{-8}$ seconds). The radiation is termed unpolarized, even though it *is* polarized when viewed within this short time period. Natural light is neither totally polarized nor totally unpolarized; rather, we speak of partial polarization.

7.6 DESCRIPTION OF POLARIZATION STATES

7.6.1 Jones Vector

The components of the electric field in the two orthogonal directions may be used to represent a vector with complex components. This is called a *Jones vector*, which is useful *only* for completely polarized light.

$$\begin{aligned} \mathbf{E} &= \operatorname{Re}\{\mathbf{E}_0 e^{i[kz - \omega t]}\} = \left[\operatorname{Re}\{E_x e^{i[kz - \omega t]}\}, \operatorname{Re}\{E_y e^{i(kz - \omega t - \delta)}\} \right] \\ &= \operatorname{Re}\left\{ [E_x, E_y e^{-i\delta}] e^{i[kz - \omega t]} \right\} \\ \Rightarrow \text{Jones Vector } \mathcal{E} &= \begin{bmatrix} E_x \\ E_y e^{-i\delta} \end{bmatrix} \end{aligned}$$

Examples:

1. Plane-polarized light along x -axis

$$\mathcal{E} = \begin{bmatrix} E_0 \\ 0 \end{bmatrix}$$

2. Plane-polarized light along y -axis:

$$\mathcal{E} = \begin{bmatrix} 0 \\ E_0 \end{bmatrix}$$

3. Plane-polarized light at angle θ to x -axis:

$$\mathcal{E} = \begin{bmatrix} E_0 \cos[\theta] \\ E_0 \sin[\theta] \end{bmatrix}$$

4. RHCP

$$\begin{aligned} \mathbf{E} &= \hat{\mathbf{x}}E_0 \cos[kz - \omega t] + \hat{\mathbf{y}}E_0 \sin[kz - \omega t] \\ &= \hat{\mathbf{x}}E_0 \cos[kz - \omega t] + \hat{\mathbf{y}}E_0 \cos\left[kz - \omega t - \frac{\pi}{2}\right] \\ &= \operatorname{Re} \left\{ \begin{bmatrix} E_0 \\ E_0 e^{-i\frac{\pi}{2}} \end{bmatrix} e^{i[kz - \omega t]} \right\} \implies \mathcal{E} = \operatorname{Re} \left\{ E_0 \begin{bmatrix} 1 \\ e^{-i\frac{\pi}{2}} \end{bmatrix} e^{i[kz - \omega t]} \right\} \end{aligned}$$

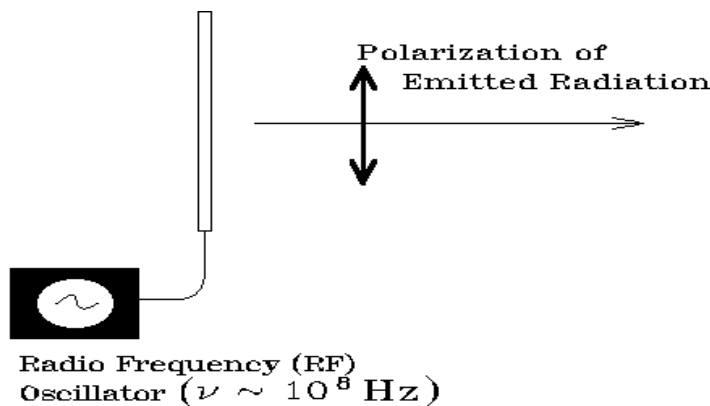
Other representations of the state of polarization are available (*e.g.*, Stokes' parameters, coherency matrix, Mueller matrix, Poincare sphere). They are more complicated, and hence more useful, *i.e.*, they can describe partially polarized states. For more information, see (for example), *Polarized Light* by Shurcliff.

7.7 GENERATION OF POLARIZED LIGHT

7.7.1 SELECTIVE EMISSION:

If all emitting elements of a source (*e.g.*, electrons in a bulb filament), vibrate in the same direction, the radiated light will be polarized in that direction. This is difficult at optical frequencies ($\nu \simeq 10^{15} \text{ Hz}$), but is easy at radio or microwave frequencies ($\nu \lesssim 10^8 \text{ Hz}$) by proper antenna design:

Rather than generating polarized light at the source, we can obtain light of a selected polarization from natural light by removing unwanted states of polarization.



“Light” (electromagnetic radiation) emitted by a “dipole” radiator is polarized in the direction of motion of the emitting electrons (vertical, in this case).

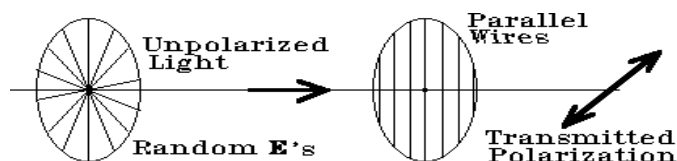
7.7.2 SELECTIVE TRANSMISSION/ABSORPTION

Many natural crystals and manmade materials affect the two orthogonal polarizations differently, due to an anisotropy (nonuniformity) in their crystalline structure. Such materials are called *dichroic*. Many crystals (*e.g.*, calcite) divide a nonpolarized light wave into two components with orthogonal polarizations. One is called the *ordinary ray*, because it obeys Snell's law for refraction. The second, or *extraordinary ray*, does not obey Snell. By dividing the incoming natural light into two beams in such a crystal, we can select one of the two polarizations.

A manmade device for selecting a state of polarization by selective absorption is *polaroid*. This operates like the microwave-polarizing skein of wires:

The wires are parallel to the y -axis in this case. Radiation incident on the wires drives the free electrons in the wires in the direction of polarization of the radiation. The electrons driven in the y -direction along the surface of the wire and strike other such electrons, thus dissipating the energy in thermal collisions. What energy that is reradiated by such electrons is mostly directed back toward the source (reflected). The x -component of the polarization is not so affected, since the electrons in the wire are constrained against movement in that direction. The x -component of the radiation therefore passes nearly unaffected.

Common polaroid sheet acts as a skein of wires for optical radiation. It is made from clear polyvinyl acetate which has been stretched in one direction to produce long chains of hydrocarbon molecules. The sheet is then immersed in iodine to supply lots of free electrons.

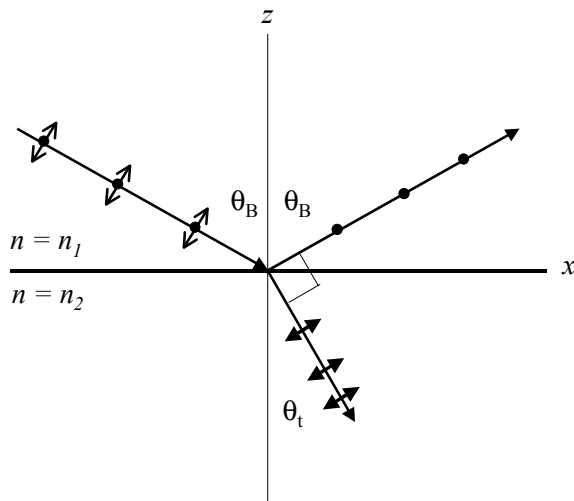


*Polarization by “skein of wires” – the radiation polarized parallel to the direction of the wires in the skein is absorbed, so the radiation polarized **perpendicular to the wires** is transmitted.*

7.7.3 GENERATING POLARIZED LIGHT BY REFLECTION, BREWSTER'S ANGLE

H§8.6

The two polarizations of light reflected from an interface between two different dielectric media (*i.e.*, media with different real refractive indices) see the same configuration of the interface only with normal incidence (*i.e.*, the light is incident perpendicular to the surface). Thus the two polarizations must be identically reflected. However, if the light is incident obliquely, one polarization “sees” the bound electrons of the surface differently and therefore is reflected differently. The reflected wave is polarized to some extent; the amount of polarization depends on the angle of incidence and the index of refraction n . The polarization mechanism is simply pictured as a *forced electron oscillator*. The bound electrons in the dielectric material are driven by the incident oscillating electric field of the radiation $\mathbf{E}e^{i(kz_0 \pm \omega t)}$, and hence vibrate at frequency $\nu = \frac{\omega}{2\pi}$. Due to its acceleration, the vibrating electron reradiates radiation at the same frequency ν to produce the reflected wave. The state of polarization of the reflected radiation is a function of the polarization state of the incident wave, the angle of incidence, and the indices of refraction on either side of the interface. If the reflected wave and the refracted wave are orthogonal (*i.e.*, $\theta_i + \theta_r = 90^\circ \implies \theta_r = \frac{\pi}{2} - \theta_i$), then the reflected wave is completely plane polarized parallel to the surface (and thus polarized perpendicular to the plane of incidence). This results because the electrons driven *in* the plane of the incidence will not emit radiation at the angle required by the law of reflection. This angle of complete polarization is called *Brewster's Angle* θ_B .



Polarization at of reflected light at Brewster's angle. The sum of the incident and refracted angle is $90^\circ = \frac{\pi}{2}$. Thus $\theta_B + \theta_t = \frac{\pi}{2} \implies \theta_t = \frac{\pi}{2} - \theta_B$.

From Snell's law:

$$n_1 \sin [\theta_1] = n_2 \sin [\theta_2]$$

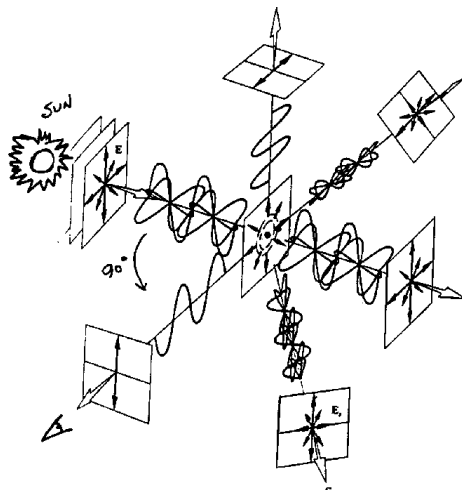
At Brewster's angle,

$$\begin{aligned} n_1 \sin [\theta_B] &= n_2 \sin \left[\frac{\pi}{2} - \theta_B \right] \\ &= n_2 \left(\sin \left[\frac{\pi}{2} \right] \cos [\theta_B] - \cos \left[\frac{\pi}{2} \right] \sin [\theta_B] \right) \\ &= +n_2 \cos [\theta_B] \\ n_1 \sin [\theta_B] &= n_2 \cos [\theta_B] \\ \implies \frac{n_2}{n_1} &= \frac{\sin [\theta_B]}{\cos [\theta_B]} = \tan [\theta_B] \\ \implies \theta_B &= \tan^{-1} \left[\frac{n_2}{n_1} \right] \end{aligned}$$

If $n_1 = 1$ (air) and $n_2 = 1.5$ (glass), then $\theta_B \simeq 56.3^\circ$. For incident angles larger than about 56° , the reflected light is plane polarized parallel to the plane of incidence. If the dense medium is water ($n_2 = 1.33$), then $\theta_B \simeq 52.4^\circ$. This happens at the interface with any dielectric. The reflection at Brewster's angle provides a handy means to determine the polarization axis of a linear polarizer – just look through the polarizer at light reflected at a steep angle.

7.7.4 POLARIZATION BY SCATTERING

Light impinging on an air molecule drives the electrons of the molecule in the direction of vibration of the electric field vector. This motion causes light to be *reradiated* in a dipole pattern; *i.e.*, no light is emitted along the direction of electron vibration. If we look at scattered light (*e.g.*, blue sky) at 90° from the source, the light is completely linearly polarized. Note that if the light is multiply scattered, as in fog, each scattering disturbs the state of polarization and the overall linear state is perturbed into unpolarized radiation.



Scattering of sunlight by atmospheric molecules.

7.8 BIREFRINGENCE – DOUBLE REFRACTION

H§8.4

Some materials have different refractive indices for orthogonal polarizations of incident light; such materials are called *birefringent*. This is due to some anisotropy (nonuniformity) of the material (just as for dichroics). Some natural materials, such as calcite, and many synthetics exhibit birefringence. The two indices of refraction are sometimes denoted n_f and n_s for *fast* and *slow* axes, where $n_f < n_s$. They are also denoted n_o and n_e for *ordinary* and *extraordinary* axes. The ordinary ray obeys Snell's law; the extraordinary ray does not.

Examples:

Refractive indices along the fast and slow axes at $\lambda = 589.3nm$

Material	n_s	n_f
Calcite ($CaCO_3$)	1.6584	1.4864
Crystalline Quartz (SiO_2)	1.5534	1.5443
Ice (crystalline H_2O)	1.313	1.309
Rutile (TiO_2)	2.903	2.616
Sodium Nitrate ($SiNO_3$)	1.5854	1.3369

The wavelength of light in a medium is $\lambda' = \frac{\lambda}{n}$, so light along the two polarization directions have different wavelengths:

$$\lambda'_s = \frac{\lambda}{n_s} < \lambda'_f = \frac{\lambda}{n_f}$$

7.8.1 PHASE DELAYS IN BIREFRINGENT MATERIAL – WAVE PLATES

Consider light incident on a birefringent material of thickness d . The electric field as a function of distance z and time t is:

$$\mathbf{E}[z, t] = (\hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y) e^{i(kz - \omega t)}.$$

At the input face of the material ($z = 0$) and the output face ($z = d$), the fields are:

$$\begin{aligned} \mathbf{E}[z = 0, t] &= (\mathbf{x}E_x + \mathbf{y}E_y) e^{-i\omega t} \\ \mathbf{E}[z = d, t] &= (\mathbf{x}E_x + \mathbf{y}E_y) e^{i(kd - \omega t)} \end{aligned}$$

If $n_x = n_s > n_y = n_f$, then $\lambda_f > \lambda_s$ and :

$$k_s = k_x = \frac{2\pi n_s}{\lambda} > k_f = k_y = \frac{2\pi n_f}{\lambda}$$

The field at the output face ($z = d$) is therefore:

$$\begin{aligned} \mathbf{E}[d, t] &= \left[\hat{\mathbf{x}}E_x e^{\frac{i(2\pi d(n_s))}{\lambda}} + \hat{\mathbf{y}}E_y e^{\frac{i(2\pi d(n_f))}{\lambda}} \right] e^{-i\omega t} \\ &= \left[\hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y e^{\frac{2\pi i}{\lambda} d(n_f - n_s)} \right] e^{i\left(\frac{2\pi d(n_s)}{\lambda} - \omega t\right)} \end{aligned}$$

By defining a constant phase term $\delta \equiv \frac{2\pi}{\lambda} d(n_f - n_s)$, the electric field at the output face of the birefringent material can be expressed as:

$$\mathbf{E}[d, t] = (\hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y e^{i\delta}) e^{i\left(\frac{2\pi d(n_s)}{\lambda} - \omega t\right)}$$

On emergence from the material, the y -component of the polarization has a different phase than the x -component; the phase difference is δ .

Example:

$\delta = \frac{\pi}{2} \implies (n_f - n_s)d = -\frac{\lambda}{4}$, and there is a phase difference of one quarter wavelength between the polarizations of the x - and the y -components of the wave. This is a *quarter-wave plate*. The required thickness d of the material is:

$$d = \frac{\lambda}{4(n_s - n_f)}$$

And the emerging field is:

$$\mathbf{E}[d, t] = \left[\hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y e^{\frac{i\pi}{2}} \right] e^{i(k_s d - \omega t)}$$

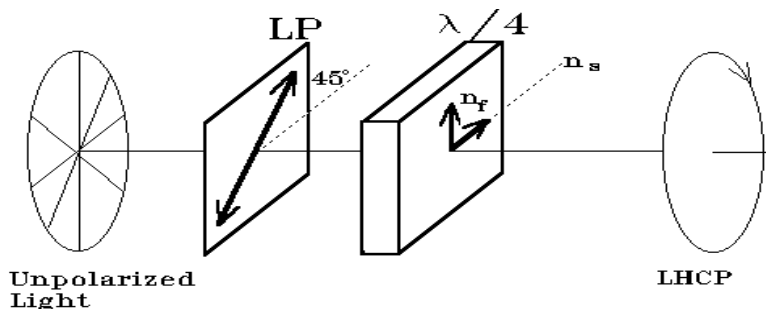
If $E_x = E_y$, (*i.e.*, the incident wave is linearly polarized @ 45° to the x -axis), then the emerging wave is circularly polarized. This is the principle of the circular polarizer.

Example:

If $\delta = \pi$, $d = \frac{\lambda}{2(n_s - n_f)}$, and the relative phase delay is 180° . Such a device is a *half-wave plate*. If the incident light is linearly polarized oriented midway between the fast and slow axes, the plane of polarization of the exiting linearly polarized light is rotated by 90° .

7.8.2 CIRCULAR POLARIZER:

A circular polarizer is a sandwich of a linear polarizer and a $\frac{\lambda}{4}$ plate, where the polarizing axis is oriented midway between the fast and slow axes of the quarter-wave plate. The LP ensures that equal amplitudes exist along both axes of the quarter-wave plate, which delays one of the components to create circularly polarized light. Light incident from the back side of a circular polarizer is not circularly polarized on exit; rather it is linearly polarized. A circular polarizer can be recognized and properly oriented by placing it on a reflecting object (*e.g.*, a dime). If the image of the coin is dark, the polarizer has the linear polarizer on top. This is because the handedness of the light is changed on reflection; the light emerging from the $\frac{\lambda}{4}$ plate is now linearly polarized perpendicular to the axis of the LP and no light escapes.



A circular polarizer is a sandwich of a linear polarizer and a quarter-wave plate.

7.9 Critical Angle – Total Internal Reflection

From Snell, we have the relation:

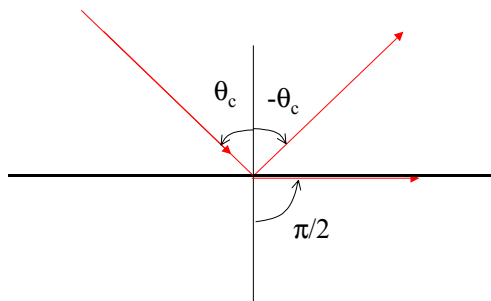
$$\sin[\theta_2] = \frac{n_1}{n_2} \sin[\theta_1]$$

If $n_1 > n_2$ then a specific angle θ_1 satisfies the condition:

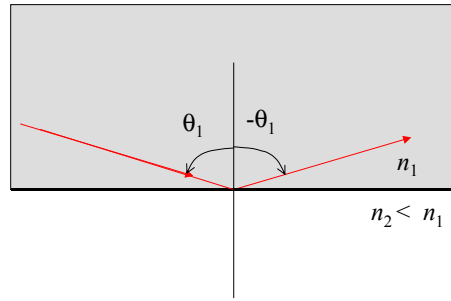
$$\frac{n_1}{n_2} \sin[\theta_1] = 1 \implies \sin[\theta_1] = \frac{n_2}{n_1} < 1 \implies \theta_2 = \frac{\pi}{2}$$

which means that the outgoing ray is refracted parallel to the interface (“surface”). The incident angle θ_1 that satisfies this condition is the *critical angle* θ_c

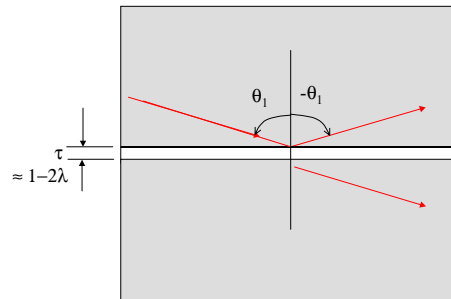
$$\theta_c = \sin^{-1} \left[\frac{n_2}{n_1} \right]$$



For crown glass with $n_d = 1.52$, the critical angle is $\sin^{-1} \left[\frac{1}{1.52} \right] \simeq 0.718$ radians $\simeq 41^\circ$. For a common flint glass with $n_d = 1.70$, then $\theta_c \simeq 0.629$ radians $\simeq 36^\circ$. If the incident angle $\theta_1 > \theta_c$ and $n_1 > n_2$ (e.g., the first medium is glass and the second is air), then no real-valued solution for Snell’s law exists, and there is no refracted light. This is the well-known phenomenon of *total internal reflection* – all of the incident light is reflected at the interface.



This may be analyzed rigorously by applying Maxwell's equations to show that the refracted angle θ_2 is *complex valued*, not real, and that the electromagnetic field is attenuated exponentially as it crosses the interface. In other words, the electric field decays so rapidly across the boundary that no energy can flow across the boundary, and hence no light escapes. However, we can “frustrate” the total internal reflection by placing another medium (such as another piece of glass) within a few light wavelengths of the interface. If close enough to the boundary, then some electric field can get into the second glass and a refracted wave “escapes”.



Schematic of “frustrated total internal reflection”: some energy can “jump” across a small gap between two pieces of glass even though the incident angle exceeds the critical angle. As the width τ of the gap increases, then the quantity of energy coupled across the gap decreases very quickly.