

Chapter 6

Vector Calculus

References: Skilling, *Fundamentals of Electric Waves* § 2; Lorrain & Corson, *Intro. to Electromagnetic Fields and Waves*, § 1; Hecht, *Optics*, § 3.2, Appendix 1

In 1864, James Clerk Maxwell published a paper on the dynamics of electromagnetic fields, in which he collected four previously described equations which relate electric and magnetic forces, modified one, and combined them to demonstrate the true nature of light waves. The four equations are now collected into a group that bears his name.

To interpret the four Maxwell equations, we must first understand some concepts of *differential vector calculus*, which seems intimidating but is really just an extension of normal differentiation applied to scalar and vector fields. For our purposes, a *scalar field* is a description of scalar values in space (one or more spatial dimensions). One example of a scalar field is the temperature distribution in the air throughout the atmosphere. Obviously, a single number is assigned to each point in the space. On the other hand, a *vector field* defines the values of a vector quantity throughout a volume. For example, the vector field of wind velocity in the atmosphere assigns a three-dimensional vector to each point in space. In my notation, scalar quantities are denoted by normal-face type and vectors (usually) by overscored bold face, *e.g.*, $f[x, y, z]$ and $\mathbf{g}[x, y, z]$ describe scalar and vector fields, respectively. Unit vectors (vectors with unit magnitude, also called unit length) are indicated by bold-faced characters topped by a caret,

In vector calculus, spatial derivatives are performed on vector AND scalar fields to derive other vector or scalar fields. The first-order differential operator ∇ (called “*del*”) has three components:

$$\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right]$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are unit vectors in the x , y , and z directions respectively. The del-operator may be applied to a scalar field to create a 3-D vector field (the *gradient* operation), or to a vector field to create a scalar field (the *divergence*), or to a vector field to create a 3-D vector field (the *curl*). The first two operations may be generalized to operate on or generate 2-D vectors, whereas the curl is defined only for 3-D vector fields.

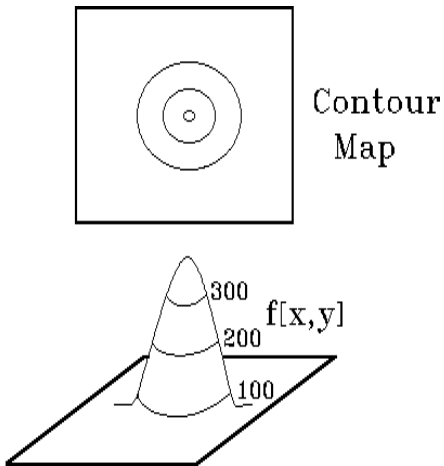
6.1 GRADIENT

Derives a Vector Field ∇f from a Scalar Field f

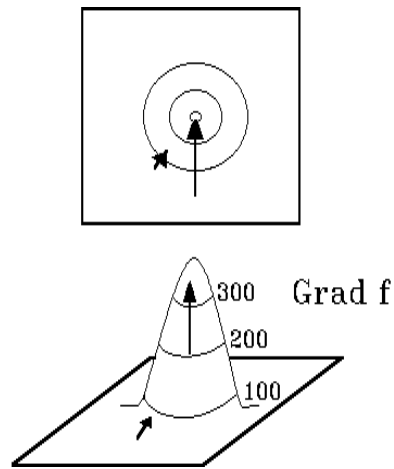
Applying ∇ to a scalar field $f[x, y, z]$ with three dimensions (such as the temperature of air at all points in the atmosphere) generates a field of 3-D vectors which describes the *spatial rate-of-change* of the scalar field, *i.e.*, the gradient of the temperature at each point in the atmosphere is a vector that describes the direction and magnitude of the change in air temperature. In the 2-D case where the scalar field describes the altitude of landform topography, the gradient vector is the size and direction of the maximum slope of the landform.

$$\nabla f[x, y, z] \equiv \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] = \hat{\mathbf{x}} \frac{\partial f}{\partial x} + \hat{\mathbf{y}} \frac{\partial f}{\partial y} + \hat{\mathbf{z}} \frac{\partial f}{\partial z} = \text{a vector}$$

As implied by its name, the gradient vector at $[x, y, z]$ points “uphill” in the direction of maximum rate-of-change of the field; the magnitude of the gradient $|\nabla f|$ is the slope of the scalar field. x :



Scalar field represented as contour map and as 3-D display.



Gradient of the scalar field is a vector field. At each coordinate $[x, y]$, the vector points “uphill” and its length is equal to the slope.

6.2 DIVERGENCE

Derives a Scalar Field $\nabla \cdot \mathbf{g}$ from a Vector Field \mathbf{g}

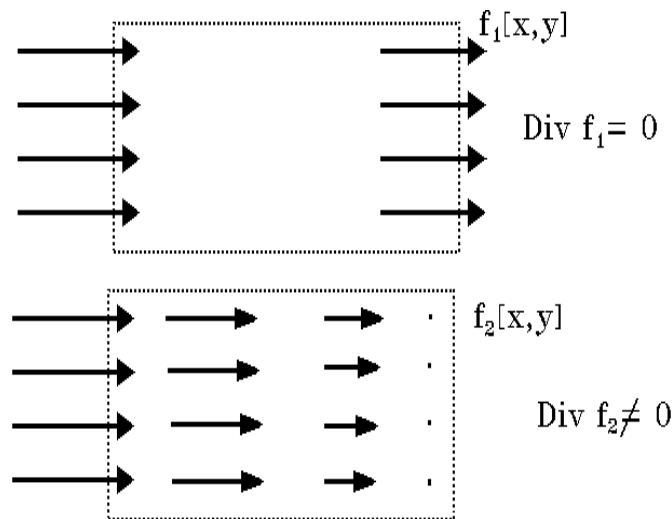
$$\begin{aligned} \nabla \cdot \mathbf{g}[x, y, z] &\equiv \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \cdot [g_x, g_y, g_z] \\ &= \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z} = a \text{ scalar} \end{aligned}$$

The divergence at each point in a vector field is a number that describes the total spatial rate-of-change, such as the total outgoing vector *flux* per unit volume. Consider a vector field \mathbf{g} and an infinitesimal surface “element” which is described by the normal vector $d\mathbf{a}$ directed *outward* from the volume. The flux F of the vector field through the surface element is the scalar (“dot”) product of the field with the normal to the surface:

$$\begin{aligned} \mathbf{g} \bullet d\mathbf{a} &= dF \\ \implies F &= \int_{\text{surface area}} \mathbf{g} \bullet d\mathbf{a} \end{aligned}$$

The divergence of the field describes total flux through the surface area A enclosing the volume. Unless the volume encloses a net “source” or “sink” of the vector field (a point from which the vector field “diverges” or “converges”), then the divergence must be zero:

$$\begin{aligned} \nabla \bullet \mathbf{g} &= \int_{\text{surface area}} \mathbf{g} \bullet d\mathbf{a} \\ &= 0 \text{ if no “source” or “sink” of vector field in volume} \end{aligned}$$



A vector field with nonzero divergence has a disparity between input and output flux

6.3 CURL

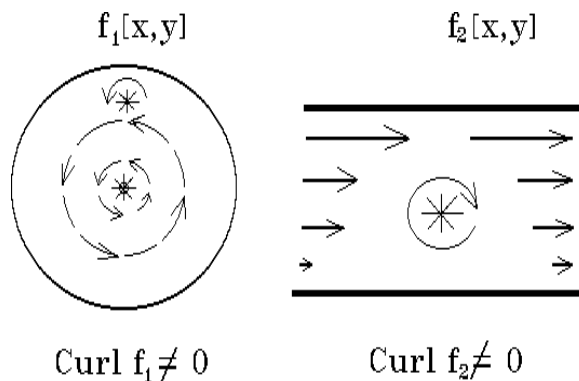
Derives a 3-D VectorField from a 3-D Vector Field g

The curl of a vector field describes a spatial nonuniformity of the 3-D vector field $\mathbf{g}[x,y,z]$.

$$\begin{aligned} \nabla \times \mathbf{g}[x, y, z] &= \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ g_x & g_y & g_z \end{bmatrix} \\ &= \hat{\mathbf{x}} \left(\frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z} \right) + \hat{\mathbf{y}} \left(\frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x} \right) + \hat{\mathbf{z}} \left(\frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y} \right) = a \text{ vector} \end{aligned}$$

To visualize curl, imagine a vector field that describes motion of a fluid (*e.g.*, water or wind). If a paddle-wheel placed in the fluid does not revolve, the field has no curl. If the wheel does revolve, the curl is nonzero. The direction of the curl vector is that of the axis of the paddlewheel when

the rotation is maximized and its magnitude is that rotation rate. The algebraic sign of the curl is determined by the direction of rotation (clockwise \implies positive curl). The paddle will rotate only if the vector field is spatially nonuniform. Note that some points in the field can have zero curl while others have nonvanishing curl. Both vector fields shown in the examples of divergence have zero curl, since a paddle wheel placed at any point in either field will not rotate.



Two vector fields with nonzero curl. The “paddlewheel” rotates in both cases.

6.4 Maxwell’s Equations

By 1864, much was known about electric and magnetic effects on materials. Faraday had discovered that a time-varying magnetic field (such as from a moving magnet) can generate an electric field, and Ampere demonstrated the corresponding effect that a time-varying electric field (as from a moving electric charge) produces a magnetic field. Both electric and magnetic fields were known to be vectors that could vary in time and space: $\mathbf{E}[x,y,z,t]$ and $\mathbf{B}[x,y,z,t]$. The electric field \mathbf{E} has units of Volts/meter, while the magnetic field \mathbf{B} is measured in Tesla or Gauss:

$$1 \text{ Tesla} = 10^4 \text{ Gauss} = 1 \frac{\text{Weber}}{\text{meter}} = 1 \frac{\text{Newton}}{\text{Amp} - \text{meter}}$$

Two other vector fields are required when describing propagation of electromagnetism through matter: the electric displacement \mathbf{D} and the magnetic field intensity \mathbf{H} . \mathbf{D} defines the total electric field in a material, which is the sum of the external field \mathbf{E} and any local field \mathbf{P} generated within the matter due to polarization from the incident \mathbf{E} field. \mathbf{H} is a similar construct for magnetic fields. \mathbf{E} and \mathbf{D} , and \mathbf{B} and \mathbf{H} are related by the so-called *constitutive equations*:

$$\begin{aligned} \mathbf{D} &= \epsilon \mathbf{E} \\ \mathbf{B} &= \mu \mathbf{H} \end{aligned}$$

where ϵ is the *electric permittivity* of the material and μ is its *magnetic permeability*. These respectively describe how well the electric and magnetic fields penetrate matter. Since we will consider propagation of light only in vacuum, we can ignore the fields \mathbf{D} and \mathbf{H} in matter. In vacuum, μ and ϵ are denoted μ_0 and ϵ_0 and both are set to unity in CGS units. In MKS units, the quantities are:

$$\begin{aligned} \mu_0 &= 4\pi \cdot 10^7 \text{ newton/amp}^2 \\ \epsilon_0 &= 8.85 \cdot 10^{-12} \text{ farad/meter.} \end{aligned}$$

As is true for the refractive index n , the permittivity and permeability in matter are larger than in vacuum, $\epsilon > \epsilon_0$, and $\mu > \mu_0$. In fact (though we won’t discuss it in detail), ϵ and μ in fact *determine*

v and n via:

$$v = \frac{1}{\sqrt{\mu\epsilon}}$$

$$n = \frac{c}{v} = \frac{\sqrt{\mu\epsilon}}{\sqrt{\mu_0\epsilon_0}}$$

Maxwell collected the four differential equations relating the electric vector field \mathbf{E} and the magnetic vector field \mathbf{B} listed below and solved them to derive the character of electromagnetic waves.

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon} \text{ Gauss' Law for Electric Fields, Coulomb's Law} \\ \nabla \cdot \mathbf{B} &= 0, \text{ Gauss' Law for Magnetic Fields} \\ -\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times \mathbf{E}, \text{ Faraday's Law of Magnetic Induction} \\ +\epsilon \frac{\partial \mathbf{E}}{\partial t} &= \nabla \times \frac{\mathbf{B}}{\mu} - \mathbf{J}, \text{ Ampere's Law}\end{aligned}$$

The scalar ρ is the electric charge density [*coulombs/m³*] and the vector field \mathbf{J} is the electric current density [*amps/m³*]. These source terms only are nonzero in media, such as copper wire. If we consider the propagation of light only in a vacuum, neither electric charges nor conductors are present and both source terms vanish.

The definition of curl may be used to rewrite the four vector equations of Maxwell as eight scalar equations:

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \text{ Gauss' Law for } \mathbf{E} \quad (1)$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \text{ Gauss' Law for } \mathbf{B} \quad (2)$$

$$-\frac{\partial B_x}{\partial t} = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \text{ Faraday's Law} \quad (3)$$

$$-\frac{\partial B_y}{\partial t} = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \quad (4)$$

$$-\frac{\partial B_z}{\partial t} = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \quad (5)$$

$$+\mu\epsilon \frac{\partial E_x}{\partial t} = \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \text{ Ampere's Law} \quad (6)$$

$$+\mu\epsilon \frac{\partial E_y}{\partial t} = \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \quad (7)$$

$$+\mu\epsilon \frac{\partial E_z}{\partial t} = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \quad (8)$$

In the general case, the electric field and magnetic fields can have the form:

$$\begin{aligned}\mathbf{E}[x, y, z, t] &= \hat{\mathbf{x}}E_x[x, y, z, t] + \hat{\mathbf{y}}E_y[x, y, z, t] + \hat{\mathbf{z}}E_z[x, y, z, t] \\ \mathbf{B}[x, y, z, t] &= \hat{\mathbf{x}}B_x[x, y, z, t] + \hat{\mathbf{y}}B_y[x, y, z, t] + \hat{\mathbf{z}}B_z[x, y, z, t]\end{aligned}$$

We will now solve these equations for a single specific case: an infinite plane electric field wave propagating in vacuum toward $z = +\infty$. The locus of points of constant phase (often called a *wavefront*) of a plane wave is (obviously) a plane. The electric field \mathbf{E} has no variation along x or y at a particular value of z , but can vary with z ; this variation will be shown to be sinusoidal. This constraint affects the derivatives of the components of the electric field:

$$\frac{\partial E_x}{\partial x} = \frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x} = \frac{\partial E_y}{\partial y} = \frac{\partial E_z}{\partial x} = \frac{\partial E_z}{\partial y} = 0 \quad (9)$$

and the 4-D vector field $\mathbf{E}[x,y,z,t]$ can be written as:

$$\mathbf{E}[x, y, z, t] = \mathbf{E}[z, t] = E_x[z, t] + E_y[z, t] + E_z[z, t] \quad (10)$$

From (9) and Gauss' law for electric fields (1), we find that:

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \implies \frac{\partial E_z}{\partial z} = 0 \quad (11)$$

Since the derivative of E_z with respect to z vanishes, then The z -component of the electric field E_z is an arbitrary constant, which we select to be 0:

$$E_z[x, y, z] = 0 \quad (12)$$

Therefore, the electric field is now expressable as:

$$\mathbf{E}[x, y, z] = E_x[z, t] + E_y[z, t] \quad (13)$$

i.e., the only existing electric field is transverse to z ! This alone is a significant result. We can simplify *eq.* (13) by rotating the coordinate system about the z axis such that \mathbf{E} is aligned with the x -axis so that

$$E_y[z, t] = 0 \text{ by assumption} \quad (14)$$

The expression for the electric field is:

$$\mathbf{E}[x, y, z] = E_x[z, t] \quad (15)$$

Given the expression for $\mathbf{E}[x,y,z,t]$, we can substitute these results into Faraday's Law (eqs. 3,4,5) to find the magnetic field:

$$-\frac{\partial B_x}{\partial t} = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0 - \frac{\partial E_y}{\partial z} \implies \frac{\partial B_x}{\partial t} = \frac{\partial E_y}{\partial z} = 0 \implies B_x[t] \text{ is constant} \quad (3)$$

$$-\frac{\partial B_y}{\partial t} = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = \frac{\partial E_x}{\partial z} \implies \frac{\partial B_y}{\partial t} = -\frac{\partial E_x}{\partial z} \quad (4)$$

$$-\frac{\partial B_z}{\partial t} = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \implies B_z[t] \text{ is constant} \quad (5)$$

We can arbitrarily set the constant term $B_z = 0$, so the only remaining equation is:

$$-\frac{\partial B_y}{\partial t} = \frac{\partial E_x}{\partial z} \quad (4)$$

which says that the *time* derivative of the magnetic field B_y is equal to to the negative of the *space* derivative of E_x . We can now find a relation between B_y and E_x by standard solution techniques of differential equations. Assume that: \mathbf{E} is a vector field that varies sinusoidally with z :

$$\begin{aligned} \mathbf{E}[x, y, z, t] &= \hat{\mathbf{x}} E_x[z, t] = \hat{\mathbf{x}} E_0 \cos[kz - \omega t] \\ \implies \frac{\partial E_x}{\partial z} &= -kE_0 \sin[kz - \omega t] \\ \implies -B_y[z, t] &= \int \frac{\partial E_x}{\partial z} dt = -(-kE_0) \int \sin[kz - \omega t] dt \\ -B_y[z, t] &= +kE_0 \cdot \left[\frac{-\cos[kz - \omega t]}{-\omega} \right] \\ &= E_0 \frac{k}{\omega} \cos[kz - \omega t] \\ &= \frac{E_0}{\left(\frac{\omega}{k}\right)} \cos[kz - \omega t] \\ &= \frac{E_0}{v_\phi} \cos[kz - \omega t] \\ \implies \mathbf{B}[z, t] &= \hat{\mathbf{y}} \left(-\frac{E_0}{v_\phi} \cos[kz - \omega t] \right) \end{aligned}$$

where v_ϕ is the phase velocity of the electromagnetic wave

$$B_y = \frac{E_0}{v_\phi} \cos[kz - \omega t] = \frac{E_x}{v_\phi} \implies E_x = v_\phi B_y$$

Note that the only existing component of \mathbf{B} is B_y , which is *perpendicular* to E_x . Also note that the sinusoidal variations of \mathbf{E} and \mathbf{B} have the same arguments, which means that they oscillate “in phase”.

6.5 PHASE VELOCITY OF ELECTROMAGNETIC WAVES

Given the form for the plane electromagnetic wave in a vacuum, we can now use the three Ampere relations to find something else useful:

$$+\mu\epsilon \frac{\partial E_x}{\partial t} = \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \quad (6)$$

$$+\mu\epsilon \frac{\partial E_y}{\partial t} = \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \quad (7)$$

$$+\mu\epsilon \frac{\partial E_z}{\partial t} = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \quad (8)$$

Because $\mathbf{E} = \hat{\mathbf{x}}E_0$, only (6) does not vanish:

$$\begin{aligned} \mu\epsilon \frac{\partial}{\partial t} (E_0 \cos[kz - \omega t]) &= -\frac{\partial}{\partial z} \left(\frac{E_0}{v_\phi} \cos[kz - \omega t] \right) \\ \implies \mu\epsilon E_0 (\omega \sin[kz - \omega t]) &= -\frac{E_0}{v_\phi} (-k \sin[kz - \omega t]) \\ \implies \mu\epsilon \omega E_0 &= \frac{E_0 k}{v_\phi} \\ \implies \mu\epsilon &= \frac{k}{\omega v_\phi} = \frac{1}{v_\phi^2} \implies v_\phi^2 = \frac{1}{\mu\epsilon} \end{aligned}$$

$$\boxed{v_\phi = \sqrt{\frac{1}{\mu\epsilon}}}$$

In vacuum, $\mu \equiv \mu_0, \epsilon \equiv \epsilon_0, v_\phi \equiv c$, and $c = \sqrt{\frac{1}{\mu_0 \epsilon_0}}$. The permittivity and permeability of free space (vacuum) can be measured in laboratory experiments, thus allowing a calculation of the phase velocity of electromagnetic waves. Using the values for μ_0 and ϵ_0 given above, we find:

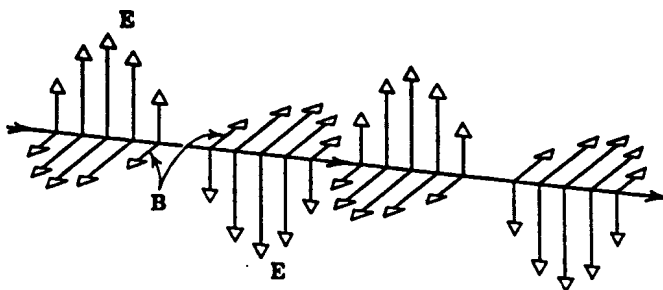
$$\begin{aligned} \mu_0 \epsilon_0 &= \left(8.85 \cdot 10^{-12} \frac{\text{coul}}{\text{v-m}} \right) \left(1.26 \cdot 10^{-6} \frac{\text{J}}{\text{amp}^2 \text{-m}} \right) \\ &= 1.11 \cdot 10^{-17} \frac{\text{coul}}{\text{v-m}} \cdot \frac{\text{J}}{\left(\frac{\text{coul}^2 \text{-m}}{\text{s}^2} \right)} \\ &= 1.11 \cdot 10^{-17} \frac{1}{\text{v-m}} \cdot \frac{\text{J-s}^2}{\text{coul-m}} \\ &= 1.11 \cdot 10^{-17} \frac{\text{s}^2}{\text{m}^2} \\ \implies c &= \sqrt{\frac{1}{\mu_0 \epsilon_0}} = 2.99 \cdot 10^8 \frac{\text{m}}{\text{s}}, \text{ which agrees well with experiment.} \end{aligned}$$

6.6 CONSEQUENCES OF MAXWELL'S EQUATIONS

1. The wave requires both electric and magnetic fields to propagate, and they copropagate, *i.e.*, they pull each other along.
2. The electric and magnetic fields of an electromagnetic wave are mutually perpendicular.
3. *The Copropagation of \mathbf{E} and \mathbf{B}* : The wave travels in a direction *mutually perpendicular* to both \mathbf{E} and \mathbf{B} , and in fact the propagation direction is defined by the direction:

$$\mathbf{S} = \mathbf{E} \times \mathbf{B} \text{ —the Poynting vector}$$

4. In vacuum, \mathbf{E} and \mathbf{B} are in-phase, which means that the phases of the sinusoidal variation of \mathbf{E} and \mathbf{B} are identical (in matter, the phases of the fields often are not so).
5. Both \mathbf{E} and \mathbf{B} travel at c , the phase velocity of the wave.
6. Energy is carried by both the electric and magnetic fields, and the magnitude of the energy $\mathcal{E} \propto E_0^2$.
7. There is no limitation on the possible frequencies of the waves, *i.e.*, $[0 \leq \omega \leq \infty]$, which implies the allowed wavelengths are in the interval $[\infty \geq \lambda \geq 0]$



Propagation of electric and magnetic fields in vacuum.

6.7 Optical Frequencies – Detector Response

The general equation for a traveling electromagnetic wave is:

$$y[z, t] = A_0 \cos[kz \mp \omega t] = A_0 \cos\left[2\pi\left(\frac{z}{\lambda} - \nu t\right)\right] = A_0 \operatorname{Re}\left\{e^{i(kz \mp \omega t)}\right\}$$

We see electromagnetic radiation with detectors, *i.e.*, devices which respond in some way to incident electromagnetic radiation. The human eye is sensitive only to visible light, *i.e.*, light with wavelengths in the range $[400\text{nm} \leq \lambda \leq 700\text{nm}]$. This is not the case for all life, however. The pit viper can see radiation emitted by humans at a wavelength of about $10\mu\text{m}$; it needs special receptors on the sides of its head to do this.

As shown in the plot of the electromagnetic spectrum, the frequencies of visible wavelengths are quite high: $\nu \simeq 10^{15}$ Hz. The temporal period of an optical wave is therefore $T = \nu^{-1} \simeq 10^{-15}$ s. Human visual receptors cannot respond fast enough to detect the periodic oscillation of the wave amplitude; we see an invariant brightness. Note that this limitation exists for all optical detectors; they all respond to the average brightness. The same is true for hearing; your ear cannot detect the variation of sound pressure due to the oscillation at frequencies above a few Hz. Because water waves have a much lower frequency, the amplitude and phase of the wave can be measured.

The average amplitude of a sinusoidal wave is:

$$\begin{aligned}\langle y[z, t] \rangle &= \frac{1}{T_d} \int_0^{T_d} y[z, t] dt \\ &= \frac{1}{T_d} \int_0^{T_d} A_0 \cos[kz - \omega t] dt \\ &= -\frac{A_0}{\omega T_d} \sin[kz - \omega t] \Big|_{t=0}^{t=T_d}\end{aligned}$$

Since $y[z, t]$ is sinusoidal, the average value of the wave will tend to zero unless T_d is smaller than the wave's temporal period. However, the intensity (squared-magnitude) of the wave does not average to zero:

$$\begin{aligned}\mathcal{E} &\propto \langle y^2[z, t] \rangle = \frac{1}{T_d} \int_0^{T_d} y^2[z, t] dt \\ &= \frac{1}{T_d} \int_0^{T_d} A_0^2 \cos^2[kz - \omega t] dt \\ &= \frac{A_0^2}{T_d} \int_0^{T_d} \cos^2[kz - \omega t] dt \\ &= \frac{A_0^2}{T_d} \cdot \frac{T_d}{2} = \frac{A_0^2}{2} \\ \implies \mathcal{E} &\propto \langle y^2[z, t] \rangle \implies \frac{A_0^2}{2} \quad \text{if } T_d \gg \nu^{-1}\end{aligned}$$

because the average value of $\cos^2[x] = \frac{1}{2}$.

Optical detectors are sensitive to time-averaged intensity, not amplitude.

