Chapter 2

Oscillations

Sources: HR §15, C§1

Before discussing the wave nature of light, it is important to review the salient characteristics of oscillations and waves from basic physics.

Oscillation – periodic variation of *any* characteristic of a physical system about some equilibrium (mean) value

e.g., position angle of a pendulum bob in a gravitational field

position of a mass on a spring

voltage across the capacitor in an LC circuit

The position angle θ of the pendulum bob and the voltage across the capacitor plates (or current in the circuit or the magnetic field generated by the inductor, ...) oscillate as functions of time. The position angle of the pendulum varies oscillates about its mean value (as measured from the vertical defined by the gravitational field)

Oscillations result from the joint presence of two forces:

1. ***Inertia***: displaces the physical quantity (*e.g.,* the position angle of the pendulum) from its equilibrium value.

2. ***Restoring (or return) force***: opposes changes in the physical quantity, acts to return it to equilibrium. The greater the deviation from equilibrium, the larger the restoring force. (acts as *negative feedback*).

Oscillations of matter can be either *transverse* or *longitudinal* (or some combination):

Longitudinal oscillation: the vectors describing the opposing forces are parallel,

e.g., a mass attached to a spring, restricted to motion toward or away from the spring;$\leftrightarrows$

Transverse oscillation: the vectors describing the two forces are not parallel,

e.g., the pendulum, where inertial force is horizontal and restoring force is vertical, unrestricted motion of mass on a helical spring

2.1 Harmonic Oscillations

The simplest oscillations are *harmonic*, *i.e.,* oscillations defined by a single sinusoidal frequency (usually the function is defined as a cosine rather than a sine because this is more compatible with complex notation). For example, consider the position angle of a pendulum in a gravitational field as a function of time:

$$y\,[t] - y_0 = A_0 \cos\{\Phi\,[t]\} = A_0 \cos(\omega t + \phi_0)$$

y is the "position" of the characteristic of the medium,*e.g.,*an angle,voltage, etc.
y_0 – equilibrium value of the characteristic;
A_0 – amplitude of the oscillation, *i.e.,* maximum displacement from equilibrium, [A] = $[y]$;
ω—angular temporal frequency of the oscillation, $[\omega] = radians/S$;
ν– temporal frequency of the oscillation, $[\nu] = cycles/S = hertz, \nu = \frac{\omega}{2\pi}$;
T– period of the oscillation, $[T] = Sec, T = \frac{1}{\nu} = \frac{2\pi}{\omega}$;
Φ– phase angle of the oscillation, $[\Phi] = radians$, (in this case, Φ is a linear function of time);
ϕ_0– initial phase of the oscillation, *i.e.,* phase angle @ $t = 0, [\phi_0] = [\Phi] = radians.$

2.2 Harmonic Oscillations – Energy Considerations

Given the equation of motion of a simple system, *e.g.,* $y\,[t] = A_0 \cos[\omega t + \phi_0]$, the velocity, acceleration, and force exerted by the system can be calculated by taking derivatives:

velocity

$$v = \frac{dy}{dt} \equiv \dot{y} = -\omega A_0 \sin[\omega t + \phi_0]$$

acceleration

$$a = \frac{d^2y}{dt^2} \equiv \ddot{y} = -\omega^2 \cos[\omega t + \phi_0]$$

inertial force

$$ma = m\,\ddot{y} = -m\left(\omega^2 A_0 \cos[\omega t + \phi_0]\right) = -m\omega^2\; y\,[t] \equiv -ky$$

the force constant of the restoring force is:

$$k \equiv m\omega^2$$

The equation for force can be transposed to $my + ky = 0$; this is the equation of motion for the simple harmonic oscillator.

From these equations, it is easy to derive the potential and kinetic energies of the harmonic oscillator:

Kinetic Energy $\mathcal{E}_k$:

$$\mathcal{E}_k[t] = \frac{1}{2}mv^2 = \frac{m}{2}\left(-\omega A_0 \sin[\omega t + \phi_0]\right)^2$$

$$\boxed{\mathcal{E}_k[t] = \frac{mA_0^2\omega^2}{2}\sin^2[\omega t + \phi_0]}$$

Potential Energy $\mathcal{E}_p$:

$$\mathcal{E}_p[t] = -\int_0^y \mathbf{F} \bullet \mathbf{ds} = -\int_0^y (-ky)\,dy = +\frac{ky^2}{2} = +\frac{m\omega^2 y^2}{2}$$

$$\boxed{\mathcal{E}_p[t] = \frac{mA_0^2\omega^2}{2}\cos^2[\omega t + \phi_0]}$$

Energy $\mathcal{E}[t]$:

$$\begin{aligned}
\mathcal{E}[t] &= \mathcal{E}_k[t] + \mathcal{E}_p[t] = \frac{mA_0^2\omega^2}{2}\sin^2[\omega t + \phi_0] + \frac{mA_0^2\omega^2}{2}\cos^2[\omega t + \phi_0] \\
&= \frac{mA_0^2\omega^2}{2}\left[\sin^2[\omega t + \phi_0] + \cos^2[\omega t + \phi_0]\right] \\
&= \frac{mA_0^2\omega^2}{2}
\end{aligned}$$

$$\boxed{\mathcal{E} = \frac{mA_0^2\omega^2}{2}}$$

1. $\mathcal{E}$ is not a function of time, *i.e.,* the total energy is constant
2. $\mathcal{E}_k$ and $\mathcal{E}_p$ are both always greater than 0.
3. $\mathcal{E} \propto A_0^2$, the energy is proportional to the *square* of the amplitude
4. $\mathcal{E} \propto \omega^2$, the energy is proportional to the *square* of the frequency: higher frequency $\Longrightarrow$ more energy
5. ω^2 is the *return force per unit displacement per unit mass.*

2.2.1 Anharmonic Oscillations

Digression:

Oscillations may also be *anharmonic*, or nonharmonic. This simply means that the characteristic of the physical system varies in a nonsinusoidal manner. For example:

The mathematical formulas for the motion and energy of the anharmonic oscillator are identical to those for the harmonic oscillator, but the derivatives and integrals are much more complicated to calculate. Fortunately, as we shall see, virtually *any* periodic function can be decomposed into the sum of harmonic functions. Recall that differentiation is *linear*, *i.e.,*

$$\text{if } f[x] = f_1[x] + f_2[x] \text{ then } \frac{df}{dx} = \frac{df_1}{dx} + \frac{df_2}{dx}$$

Therefore, the derivatives of each component may be taken separately and summed to find the derivatives of the result.

The decomposition of a function into its component frequencies is known as Fourier analysis, and will be discussed in more detail later.

2.3 Representations of Harmonic Oscillations

Since harmonic oscillators demonstrate sinusoidal motion, they may certainly be described by trigonometic functions as above.

$$y[t] = A_0 \sin[\omega t + \phi_0] = A_0 \cos\left[\frac{\pi}{2} - \omega t - \phi_0\right] = A_0 \cos\left[\omega t + \phi_0 - \frac{\pi}{2}\right]$$

where the second expression arises because $\sin\theta = \cos\left[\frac{\pi}{2} - \theta\right]$ and the last expression from the symmetry of the cosine (*i.e.,* $\cos[-\theta]) = \cos[+\theta]$). This description of oscillations is perfectly ok – it leads to all the correct results – but it can be awkward to keep a math handbook handy to recall the necessary expressions for the cosine and/or sine of sums, differences, and/or products of angles. The notation becomes even more complicated when considering the superposition (sum) of many oscillations or waves. For example, how easy is it to find the resultant of the sum of two oscillators, $y_1[t] + y_2[t]$, where $y_i = A_i \sin[\omega_i t + \phi_i]$? You can look this up to find:

$$y_1[t]+y_2[t] = \sin[\omega_1 t + \phi_1]+\sin[\omega_2 t + \phi_2] = 2\sin\left[\frac{\omega_1+\omega_2}{2}t + \frac{\phi_1+\phi_2}{2}\right]\cos\left[\frac{\omega_1-\omega_2}{2}t + \frac{\phi_1-\phi_2}{2}\right],$$

but this result is easy to derive by using complex notation, as shown in the next section.