

1 Laboratory 7: “Division-of-Wavefront” Interference, Diffraction

1.1 Theory

Recent labs on optical imaging systems have used the concept of light as a “ray” in geometrical optics to model the action of lenses. We were able to determine the locations of images and their magnifications. However, the concept of light as a “wave” also is fundamental to imaging, particularly in its manifestation in “diffraction”, which is the fundamental limitation on the action of an optical imaging system. “Interference” and “diffraction” may be interpreted as the same phenomenon, differing only in the number of sources involved (interference \implies few sources, say 2 - 10; diffraction \implies many sources, up to an infinite number). In this lab, the two sources are obtained by dividing the wave emitted by a single source by introducing two apertures into the system; this is called “division of wavefront”. In the next lab, we will divide the light by introducing a beamsplitter to create “division-of-amplitude interferometry”.

1.1.1 Interference:

We introduce interference by recalling the expression for the sum of two sinusoidal temporal oscillations of the same amplitude and different frequencies:

$$\begin{aligned} A \cos [\omega_1 t] + \cos [\omega_2 t] &= 2A \cos \left[\frac{(\omega_1 + \omega_2)}{2} t \right] \cos \left[\frac{(\omega_1 - \omega_2)}{2} t \right] \\ &= 2A \cos [\omega_{\text{mod}} t] \cos [\omega_{\text{avg}} t] \end{aligned}$$

and the travelling-wave analogue: for two plane waves propagating along the z axis:

$$A \cos [k_1 z - \omega_1 t] + A \cos [k_2 z - \omega_2 t] = 2A \cos [k_{\text{mod}} z - \omega_{\text{mod}} t] \cos [k_{\text{avg}} z - \omega_{\text{avg}} t]$$

$$k_{\text{mod}} = \frac{k_1 - k_2}{2}, \omega_{\text{mod}} = \frac{\omega_1 - \omega_2}{2}, k_{\text{avg}} = \frac{k_1 + k_2}{2}, \omega_{\text{avg}} = \frac{\omega_1 + \omega_2}{2}$$

If the dispersion is normal, the resulting wave is the product of a slow traveling wave with velocity $v_{\text{mod}} = \frac{\omega_{\text{mod}}}{k_{\text{mod}}}$ and a rapid traveling wave with velocity $v_{\text{avg}} = \frac{\omega_{\text{avg}}}{k_{\text{avg}}}$. Thus far, we have described traveling waves directed along one axis (usually z). Of course, the equations can be generalized easily to model waves traveling in any direction. Instead of a scalar angular wavenumber k , we can define the 3-D *wavevector*:

$$\underline{\mathbf{k}} = [k_x, k_y, k_z] = \hat{\mathbf{x}}k_x + \hat{\mathbf{y}}k_y + \hat{\mathbf{z}}k_z$$

which points in the direction of travel of the wave. The length of the wavevector is proportional to λ^{-1} .

$$|\underline{\mathbf{k}}| = \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{2\pi}{\lambda}.$$

Thus the equation for a traveling wave in 3-D space becomes:

$$f [x, y, z, t] = f [\underline{\mathbf{r}}, t] = A \cos [k_x x + k_y y + k_z z - \omega t] = A \cos [\underline{\mathbf{k}} \cdot \underline{\mathbf{r}} - \omega t]$$

For simplicity, we will limit ourselves to the 2-D case, where the equation of a wave is:

$$f [x, y, t] = A \cos [k_x x + k_y y - \omega t] = A \cos [\underline{\mathbf{k}} \cdot \underline{\mathbf{r}} - \omega t]$$

A 2-D or 3-D wavefront can exhibit a periodic variation in the phase $\phi [\underline{\mathbf{r}}, t] = \underline{\mathbf{k}} \cdot \underline{\mathbf{r}} - \omega t$, even if $\omega_1 = \omega_2 = \omega \rightarrow \lambda_1 = \lambda_2 = \lambda$. If light from a single source is divided into two sections by introducing

two apertures into the system, Huygens' principle indicates that the light through the two apertures will "spread" and recombine. When viewed at a single location, the two "beams" of light with the same wavelength will recombine with different wavevectors such that $|\mathbf{k}_1| = |\mathbf{k}_2| = |\mathbf{k}|$. This happens when the Cartesian sum of the components of \mathbf{k} are the same, but

$$[(k_x)_1, (k_y)_1] \neq [(k_x)_2, (k_y)_2]$$

Note that the following simple result is true only for

$$\lambda_1 = \lambda_2.$$

Light of the same wavelength (and same optical frequency) is described as *coherent*. In this case, consider the superposition of two plane waves of the same optical frequency ω , one traveling in direction \mathbf{k}_1 and one in direction \mathbf{k}_2 :

$$\begin{aligned} f_1 [x, y, z, t] &= A \cos [\mathbf{k}_1 \cdot \mathbf{r} - \omega t] \\ f_2 [x, y, z, t] &= A \cos [\mathbf{k}_2 \cdot \mathbf{r} - \omega t], \end{aligned}$$

where $\mathbf{k}_1 = [0, k_y, k_z]$ and $\mathbf{k}_2 = [0, -k_y, k_z]$, *i.e.*, the wavevectors differ only in the sign of the y -component. The wavevectors have the same length:

$$|\mathbf{k}_1| = |\mathbf{k}_2| = \frac{2\pi}{\lambda} \implies \lambda_1 = \lambda_2 \equiv \lambda.$$

The components of the wavevectors are:

$$\begin{aligned} k_z &= |\mathbf{k}| \cos [\theta] = \frac{2\pi}{\lambda} \cos [\theta] \\ k_y &= \frac{2\pi}{\lambda} \sin [\theta]. \end{aligned}$$

The superposition of the electric fields is:

$$f_1 [x, y, z, t] + f_2 [x, y, z, t] = A \{ \cos [(k_z z - \omega t) + k_y y] + \cos [(k_z z - \omega t) - k_y y] \}$$

which can be recast using the formula for $\cos [\alpha \pm \beta]$ to:

$$\begin{aligned} f_1 [x, y, z, t] + f_2 [x, y, z, t] &= 2A \cos [k_y y] \cos [k_z z - \omega t] \\ &= 2A \cos \left[\left(\frac{2\pi}{\lambda} \sin [\theta] \right) y \right] \cos [k_z z - \omega t] \end{aligned}$$

Note that there is no time dependence in the first term: this is a time-invariant pattern. Also recall that the measured quantity is the *intensity* of the pattern, which is the time average of the squared magnitude. The time-invariant term thus becomes the visible pattern:

$$\begin{aligned} |f_1 [x, y, z, t] + f_2 [x, y, z, t]|^2 &\propto 4A^2 \cos^2 \left[\frac{2\pi y}{\lambda} \sin [\theta] \right] \\ &= 4A^2 \cdot \frac{1}{2} \left(1 + \cos \left[\frac{4\pi y}{\lambda} \sin [\theta] \right] \right) 1 [y] \\ &\propto 2A^2 \cos \left[\frac{2\pi y}{D} \right], \text{ where } D \equiv \frac{\lambda}{2 \cdot \sin [\theta]} \end{aligned}$$

where the identity $\cos^2 [\beta] = \frac{1}{2} (1 + \cos [2\beta])$ has been used. The intensity pattern has a cosine form (maximum at the center) and a period proportional to λ and inversely proportional to $\sin [\theta]$. If θ is small, the period of the pattern is long. If the distance to the observation plane is large, $\sin [\theta] \simeq \frac{d}{L}$, which leads to the result that:

$$D \simeq \frac{\lambda L}{d} \rightarrow \boxed{Dd \simeq L\lambda}$$

1.1.2 Diffraction:

This lab will also investigate the diffraction patterns generated from apertures of different shapes and observed at different distances. As we have mentioned, the physical process that results in observable diffraction patterns is identical to that responsible for interference. In the latter case, we generally speak of the intensity patterns generated by light after passing through a few apertures whose size(s) generally are smaller than the distance between them.

The term *diffraction* usually is applied to the process either for a single large aperture, or (equivalently) a large number of small (usually infinitesimal) contiguous apertures.

In both cases, the patterns are most obvious when the illumination is *coherent*, which means that the phase of the sinusoidal electric fields is rigidly deterministic. In other words, knowledge of the phase of the field at some point in space and/or time determines the phase at other points in space and/or time. Coherence has two flavors: spatial and temporal. For *spatially coherent* light, the phase difference $\Delta\phi \equiv \phi_1 - \phi_2$ of the electric field measured at two different points at the same time at two points in space separated by a vector distance Δr remains constant for all times and for all such points in space. If the phase difference measured at the SAME location at two different times separated by $\Delta t \equiv t_1 - t_2$ is the same for all points in space, the light is *temporally coherent*. Light from a laser may be considered to be BOTH spatially and temporally coherent. The properties of coherent light allow phase differences of light that has traveled different paths to be made visible, since the phase difference is constant with time. In interference, the effect often results in a sinusoidal fringe pattern in space. In diffraction, the phase difference of light from different points in the same large source can be seen as a similar pattern of dark and bright fringes, though not (usually) with sinusoidal spacing.

Observed diffraction patterns from the same object usually look very different at different distances to the observation plane. If viewed very close to the aperture (in the *Rayleigh-Sommerfeld* diffraction region), then Huygens' principle says that the amplitude of the electric field is the summation (integral) of the spherical wavefronts generated by each point in the aperture. The resulting amplitude pattern may be quite complicated to evaluate. If observed somewhat farther from the aperture, the spherical wavefronts may be accurately approximated by paraboloidal wavefronts. The approximation applies in the *near field*, or the *Fresnel diffraction region*. If viewed at a large distance compared to the extent of the object, the light from different locations in the aperture may be accurately modeled as *plane waves* with different wavefront tilts. This occurs in the *Fraunhofer diffraction region*.

Fresnel Diffraction In the Fresnel diffraction region (where the distance between the object and the observation plane is small compared to the size of the object). The diffraction pattern resembles the original object with “fuzzy” or “ringing” edges. If the size of the object is increased, so will be the Fresnel diffraction pattern. If the distance between the object and the observation is increased, though still within the Fresnel diffraction region, the size of the “ringing” artifacts increases. In general, the Fresnel diffraction pattern of an object $f[x, y]$ observed at a distance z_1 “downstream” is proportional to the complicated expression:

$$g[x, y] \propto \iint_{-\infty}^{+\infty} f[x - \alpha, y - \beta] \exp \left[-\frac{i\pi(\alpha^2 + \beta^2)}{\lambda z_1} \right] d\alpha d\beta$$

which may be evaluated as the mathematical operation of “convolution” of the object pattern and a *quadratic-phase pattern* (which represents the paraboloidal shape of the individual waves).

Fraunhofer Diffraction At large distances from the object, the diffraction is in the *far field* or *Fraunhofer* diffraction region. Here, the pattern of diffracted light usually does not resemble the object at all. The size of the observed pattern varies in proportion to the *reciprocal* of the object dimension, i.e., the larger the object, the smaller the diffraction pattern. Note that increasing the size of the object also produces a brighter diffraction pattern, because more light reaches the observation plane. The mathematical relation between the shape and size of the output relative to

that of the input is a *Fourier transform*. For the same input pattern $f[x, y]$, the diffraction pattern in the Fraunhofer region has the form:

$$g[x, y] \propto \iint_{-\infty}^{+\infty} f[\alpha, \beta] \exp \left[-\frac{2\pi i(x\alpha + y\beta)}{\lambda z} \right] d\alpha d\beta$$

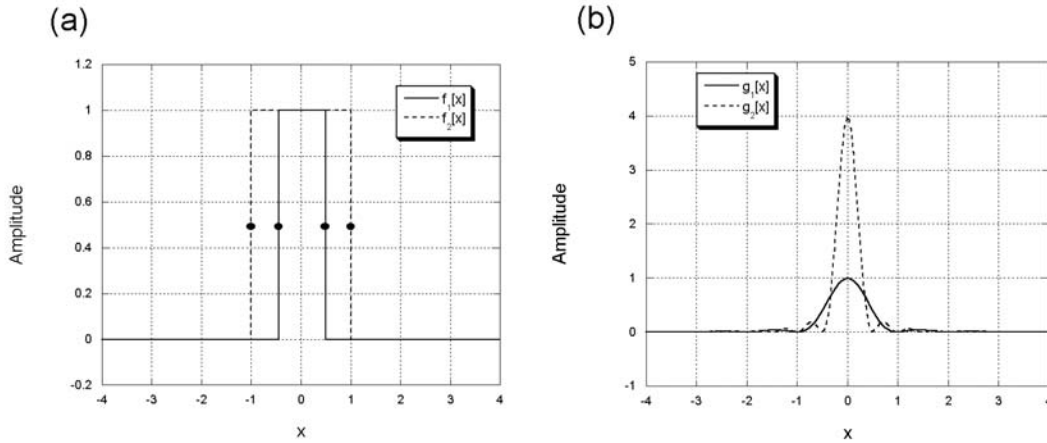
Consider the case for a simple rectangular object:

$$f[x, y] = \text{RECT} \left[\frac{x}{a}, \frac{y}{b} \right] \equiv \begin{cases} 1 & \text{if } |x| < \frac{a}{2} \text{ and } |y| < \frac{b}{2} \\ \frac{1}{2} & \text{if } |x| = \frac{a}{2} \text{ or } |y| = \frac{b}{2} \\ \frac{1}{4} & \text{if } |x| = \frac{a}{2} \text{ and } |y| = \frac{b}{2} \\ 0 & \text{if } |x| > \frac{a}{2} \text{ and } |y| > \frac{b}{2} \end{cases}$$

The integral evaluates rather easily:

$$\begin{aligned} g[x, y] &\propto \left| \iint_{-\infty}^{+\infty} \text{RECT} \left[\frac{\alpha}{a}, \frac{\beta}{b} \right] \exp \left[-\frac{2\pi i(x\alpha + y\beta)}{\lambda z} \right] d\alpha d\beta \right|^2 \\ &= \int_{y=-\frac{b}{2}}^{y=+\frac{b}{2}} \int_{x=-\frac{a}{2}}^{x=+\frac{a}{2}} \exp \left[-\frac{2\pi i x \alpha}{\lambda z} \right] \exp \left[-\frac{2\pi i y \beta}{\lambda z} \right] d\alpha d\beta = \int_{x=-\frac{a}{2}}^{x=+\frac{a}{2}} \exp \left[\left(-\frac{2\pi i x}{\lambda z} \right) \alpha \right] d\alpha \int_{y=-\frac{b}{2}}^{y=+\frac{b}{2}} \exp \left[\left(-\frac{2\pi i y}{\lambda z} \right) \beta \right] d\beta \\ &= \frac{\exp \left[\left(-\frac{2\pi i x}{\lambda z} \right) \alpha \right] \Big|_{\alpha=-\frac{a}{2}}^{\alpha=+\frac{a}{2}}}{\left(-\frac{2\pi i x}{\lambda z} \right)} \cdot \frac{\exp \left[\left(-\frac{2\pi i y}{\lambda z} \right) \beta \right] \Big|_{\beta=-\frac{b}{2}}^{\beta=+\frac{b}{2}}}{\left(-\frac{2\pi i y}{\lambda z} \right)} \\ &= \frac{\exp \left[-i \frac{\pi a x}{\lambda z} \right] - \exp \left[+i \frac{\pi a x}{\lambda z} \right]}{\left(-\frac{2\pi i x}{\lambda z} \right)} \cdot \frac{\exp \left[-i \frac{\pi y b}{\lambda z} \right] - \exp \left[+i \frac{\pi y b}{\lambda z} \right]}{\left(-\frac{2\pi i b}{\lambda z} \right)} \\ &= |a| \left(\frac{\sin \left[\frac{\pi a x}{\lambda z} \right]}{\left(\frac{\pi a x}{\lambda z} \right)} \right) \cdot |b| \left(\frac{\sin \left[\frac{\pi b y}{\lambda z} \right]}{\left(\frac{\pi b y}{\lambda z} \right)} \right) \equiv |ab| \text{SINC} \left[\frac{x}{\left(\frac{\lambda z}{a} \right)}, \frac{y}{\left(\frac{\lambda z}{b} \right)} \right] \\ g[x, y] &\propto (ab)^2 \left(\text{SINC} \left[\frac{x}{\left(\frac{\lambda z}{a} \right)}, \frac{y}{\left(\frac{\lambda z}{b} \right)} \right] \right)^2 \end{aligned}$$

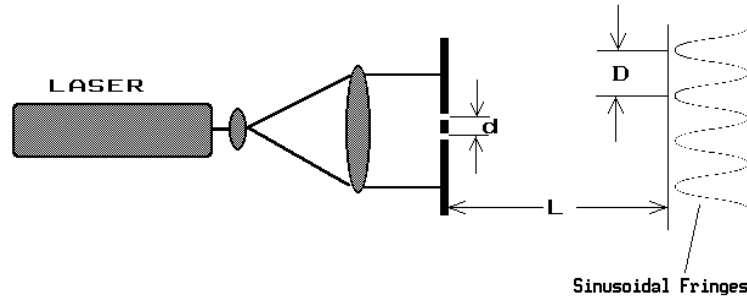
The Where the resulting ‘‘SINC’’ function has the pattern shown in the figure.



1-D model of Fresnel diffraction; (a) two 1-D rectangular objects that differ in width by a factor of two; (b) the corresponding Fresnel diffraction patterns, showing the increase in ‘‘brightness’’ and decrease in width of the diffraction pattern as the width increased.

1.2 Procedure:

1.2.1 Equipment for interference:

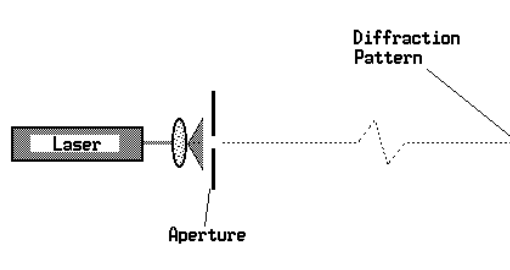


1. He:Ne Laser;
2. beam expanding lens (ideally, a “spatial filter” consisting of a microscope objective and a pinhole aperture, but a short-focal length positive lens will work);
3. collimating lens (after the expander);
4. “perfboard” (a piece of fiberboard with a regular grid of small holes used as breadboards for electronic circuits);
5. black tape (to block holes in the perfboard);
6. aluminum foil and needles, to make objects for diffraction;

1.2.2 Procedure for “Division-of-Wavefront” Interference

1. Use a piece of perfboard as the object. Perfboard has regularly spaced holes (often separated by $\frac{1}{10}$ inch) of approximately equal size. Use the black tape to cover all but one hole in the perfboard and record the image at the observation plane (either photographically or by sketch).
2. Open up an adjacent hole to create two holes separated by $d = \frac{1}{10}$ inch; record the image.
3. Record images created by two holes separated by a larger interval d_2 and by many holes. Measure the distances L , D , and d to find the approximate wavelength λ of the laser (actual $\lambda = 632.8$ nm).
4. Do a similar experiment using aluminum foil pierced by needles as the object. Record the differences between images created by a single very tiny hole and by a single larger hole. For two holes of approximately equal size, use the measured interval D of the fringe pattern, the distance L , and the known wavelength λ to calculate the separation d between the holes.

1.2.3 Equipment for Diffraction:



1. He:Ne laser
2. expanding lens (negative lens or microscope objective)
3. aluminum foil, needles, and razor blades to make targets
4. set of Metrologic transparencies
5. digital camera (optional)

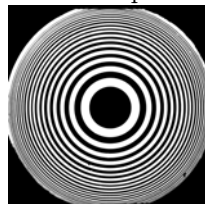
1.2.4 Procedure for Diffraction

1. Set the experimental bench as in the figure with the observing screen close to the aperture (within a foot or so) to examine the results in the *Fresnel diffraction region*. Measure and record the relevant distances. A number of apertures are available for use, including single and multiple slits of different spacings, single and multiple circular apertures, needles (both tips and eyes), razor blades, *etc.*. In addition, aluminum foil and needles are available to make your own apertures.
 - (a) Begin with a single slit, a square aperture, or a circular aperture. Note the form of the diffraction pattern. For example, sketch how its “brightness” changes with position and note the sizes and locations of any features. For a slit or circular aperture, you should note light and dark regions in the pattern; measure the positions of some maxima and minima (at least 5). Use the data to derive a scale of the pattern. Sketch the pattern noting the scale.
 - (b) Repeat the previous step with a “wider” slit or aperture. Note the difference in the results.
 - (c) Vary the distance between the screen and the diffracting object. Repeat measurements. What is the relation between the change in distance and the change in scale of the pattern? Repeat for 5 different distances where the character of the pattern remains the same.
 - (d) Repeat the procedure with a knife edge as the object. Sketch the pattern observed. You will see that the intensity distribution near the edge of the geometric shadow is not a sharp transition, but rather an undulatory pattern; a magnifying lens, microscope, or digital camera may be helpful to view the pattern, but **BE SURE THAT THE LASER LIGHT HAS BEEN ATTENUATED SUFFICIENTLY.**

2. Now observe the diffraction pattern far from the aperture (several feet away for a small aperture, a proportionally larger distance for a larger aperture) to examine *Fraunhofer diffraction*. You may “fold” the pattern with one or two mirrors or you may use a lens to “image” the pattern, i.e., to bring the image of the pattern created “a long distance away” much closer to the object. Whichever method you use, be sure to use the same setup for all measurements.
 - (a) Observe Fraunhofer diffraction from apertures of the same shape but different sizes. Measure the size of observable features and repeat this measurements using the other slits and then the other apertures. What is the influence of the physical dimension of the diffracting objects on the pattern?
 - (b) Make some of your own patterns by punching holes in aluminum foil with a needle. For example, try to make two holes close together of about the same size. Observe the pattern. Repeat after enlarging these same holes and after creating new holes somewhat farther apart. Relate the observations to the laboratory on interference by division of wavefront.
 - (c) Repeat the procedure using a periodic structure (diffraction grid or grating) as the object. Several such patterns are available in the Metrologic set with different spacings, and there are also “crossed” gratings.



- (d) Now overlay a periodic structure (grid) with a circular aperture and observe the pattern. The overlaying of the two slides produces the product of the two patterns (also called the *modulation* of one pattern by the other).
- (e) For an aperture of a known fixed (small) size, estimate the location of the boundary between the Fresnel and Fraunhofer diffraction regions. Record and justify your measurement.
- (f) Examine the image and diffraction pattern of the transparency *Albert* (Metrologic slide #18). Note the features of the diffraction pattern and relate them to the features of the transparency.
- (g) Examine the pattern generated by a Fresnel Zone Plate (Metrologic slide #13) at different distances. The FZP is a circular grating whose spacing decreases with increasing distance from the center. Sketch a side view of the FZP and indicate the diffraction angle for light incident at different distances from the center of symmetry. You might also overlap another transparency (such as a circular aperture) and the FZP and record the result. I guarantee that this result will not resemble that of part d.



- (h) If time permits, you can also find the diffraction patterns of other objects, such as the tip and/or the eye of the needle.

2 Questions

1. This experiment demonstrates that interaction of light with an obstruction will spread the light. For example, consider Fresnel diffraction of two identical small circular apertures that are separated by a distance d . How will diffraction affect the ability to distinguish the two sources? Comment on the result as lens diameter d is made smaller.
2. The Fresnel Zone Plate (Metrologic slide #13) may be viewed as a circularly symmetric grating with variable period that decreases in proportion to the radial distance from the center. It is possible to use the FZP as an imaging element (i.e., as a lens). Use the model of diffraction from a constant-period grating to describe how the FZP may be used to “focus” light in an optical imaging system. This may be useful for wavelengths (such as x rays) where imaging lenses do not exist.