

# 1 Laboratory 6: Cardinal Points of Thick Lenses and Lens Systems

## 1.1 Abstract:

In this lab, we will investigate the properties of lenses and lens systems that cannot be considered “thin”, and examine the deviations from the thin lens formula. The first exercise will be to use the concepts of “thick” lenses to determine the location of the cardinal points of four different lens systems. Once the cardinal points are located, the validity of the Gaussian formula will be verified provided that distances are measured from the principal points.

## 1.2 Theory:

In the late 1830's, Gauss discovered that the image positions and sizes in the paraxial region (*i.e.*, for rays that are always “nearly ”parallel to the optical axis) may be determined from knowledge of only four specific locations: the two *focal points* ( $F_1, F_2$ ) and the two *principal points* (sometimes labeled  $H_1$  and  $H_2$ , though other symbols are also used). We have already defined the focal points to be the locations on the optical axis of the images created from objects located at  $+\infty$  for  $F_1$  and  $-\infty$  for  $F_2$ . The principal points are pair of points (one each in object space and image space) that are images of each other with magnification  $M_T = +1$ . Note that this is not the pair of points with  $s_o = s_i = 2f$  that we often think of with unit magnification; these are images at “equal conjugates” with  $M_T = -1$ .

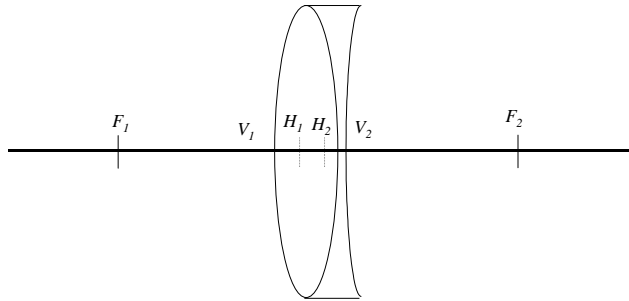


Figure 1: Schematic of the front and rear vertices  $V_1, V_2$ , the object- and image-space focal points  $F_1$  and  $F_2$ , and the object- and image-space principal points  $H_1, H_2$  of a two-element thick lens.

In 1845, Listing developed the additional concept of *nodal points*, which are the points of equal angular magnification. In other words, a ray at an angle  $\theta$  from the optical axis and directed at the front nodal point  $N_1$  will emerge from the back nodal point  $N_2$  at angle  $\theta$ . Note that for a lens in “air ” (actually, in vacuum), where both the object and image spaces have unit refractive index, the nodal points and principal points are coincident. Figure 2 shows the concept of nodal points within a thick lens.

For nonparaxial rays, the “surfaces ” of unit magnification are called the *principal planes*. The intersection of the optical axis with the front and rear surface of the lens are called the front and rear vertices respectively. Though not cardinal points, the vertices are used to specify the *focal distances*, which are the distances  $\overline{F_1V_1}$  and  $\overline{V_2F_2}$  in Figure 1. By contrast, the object- and image-space *focal lengths* are the distances  $\overline{F_1H_1}$  and  $\overline{H_2F_2}$ .

Note that the “back” or “rear” focal distance  $\overline{V_2F_2}$  is a good figure to keep in mind for the clearance required between the lens and the sensor in a real imaging system.

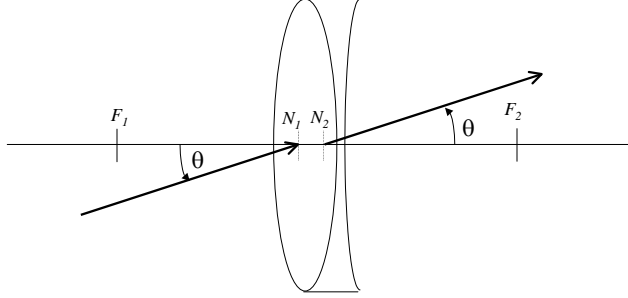


Figure 2: The concept of *nodal points*, which are the cardinal points of unit angular magnification. A ray entering the object-space nodal point  $N_1$  at angle  $\theta$  measured relative to the optical axis exits from the image-space nodal point  $N_2$  at the same angle.

The locations of the six cardinal points of an existing lens may be determined by four means: the *nodal slide*, the *foco-collimator* (or swinging collimator), the *two-magnifications method*, and the use of the *Newtonian lens formula*. Four methods are briefly described below, and the last two will be used in this experiment.

### 1.2.1 A. The Nodal Slide

The nodal slide allows the lens to be rotated about a vertical axis that may be moved longitudinally “underneath” the lens, *i.e.*, along the optical axis. If the lens is used to generate an image of a distant object (so that the incident rays may be presumed parallel) and if the vertical rotation axis is located beneath the rear nodal point, then the “image” of the distant object will not move as the lens is rotated about the vertical axis. The distance from the rear vertex to the rear nodal point can be measured by locating this point in a viewing microscope and “backing off” the longitudinal distance until the image of the rear vertex is seen. The distance  $\overline{N_2V_2}$  can be read from a distance scale. The corresponding distance between the front vertex and front nodal point is  $\overline{V_1N_1}$ .

### 1.2.2 B. The Foco-Collimator

An image of a beam of collimated light (planar wavefronts, source distance =  $+\infty$ ) is formed at the rear focal point  $f_2$  by definition. If the collimator is rotated about a vertical axis by a known small angle  $\theta$ , the image will move in a direction orthogonal to the optical axis by a distance  $h$ . The focal length is determined from:

$$f_2 = \frac{h}{\tan[\theta]} \quad (1)$$

The proof of this property is a direct result of the relationship known as the “Lagrange invariant”, which is a combination of the heights and angles of two specific rays: the *marginal ray*, which travels from the center of the object to “graze” the rim of the lens and then to the center of the image, and the *chief ray*, which travels from the edge of the object through the center of the lens to the edge of the image. As indicated by its name, the Lagrange invariant is constant at all locations in the system.

### 1.2.3 C. Two-Magnifications Method

This technique can be used to determine the focal lengths of lenses that cannot be analyzed by the nodal slide, such as microscope objectives. The apparatus consists of an optical bench, white-light source, ground glass plate, a transparent measuring scale, microscope objectives (or other test lenses), a screen, and a viewing microscope or, in our case, a CCD camera. The test lens is used to form a real image of a real object at a measured negative magnification  $m_1$ . The viewing screen is moved longitudinally by a known distance  $\Delta D$ , and the object is moved until the image is again

sharply focussed. After measuring the second magnification  $m_2$ , the back focal length is computed as follows:

$$f_2 = -\frac{\Delta D}{\Delta m} \quad (2)$$

where  $\Delta m = m_2 - m_1$ . The primary difficulty is usually measuring the transverse magnification, but one can take several data pairs  $(D, m)$  and graph the magnification versus location in image space. The inverse of the slope is the required focal length.

#### 1.2.4 D. Newtonian Lens Formula

For thick lenses, the Newtonian lens formula is expressed as:

$$x_o x_i = f_1 f_2 \quad (3)$$

where  $x_o$  is the distance from the object to the front focal point,  $x_i$  is the distance from the rear focal point to the image, and  $f_1$  and  $f_2$  are the front and back focal lengths. In other words:

$$x_o = s_1 + f_1 \quad (4a)$$

$$x_i = f_2 + s'_2 \quad (4b)$$

where  $f_1$  and  $f_2$  are the distances between the focal points and their respective principal points. For a lens in air, eq.(3) reduces to

$$x_o x_i = f^2 \implies f = \sqrt{x_o x_i} \quad (5)$$

The values of  $x_o$  and  $x_i$  can be measured for a given pair of object and image positions, after the focal points have been located by a simple measurement with a collimated beam.

### Cardinal Points of Negative Lenses

In the case of a single negative lens, rays originating at the object diverge after passing through the lens. However, a positive lens can be used in tandem with a negative lens to form a real image. For example, the focal lengths of the positive lens and of the two-lens system can be determined by the two-magnifications method, and the focal length of the negative lens is derived from these. One can, in principle, use this as a starting point to find the cardinal points of the negative lens.

## 1.3 Experimental Set-Up

We will use methods C and D to measure focal lengths, and thus to infer the focal distances, of four lens “systems”. In particular, try the following four possibilities:

1. two positive lenses from your lens kit, separated by a distance  $d$  that is smaller than either of the two focal lengths;
2. a negative and a positive lens, again separated by some “small ”(but nonzero) distance  $d$ ;
3. a single positive lens, and (if time permits);
4. a microscope objective.

The experimental set-up is shown in Figure 3.

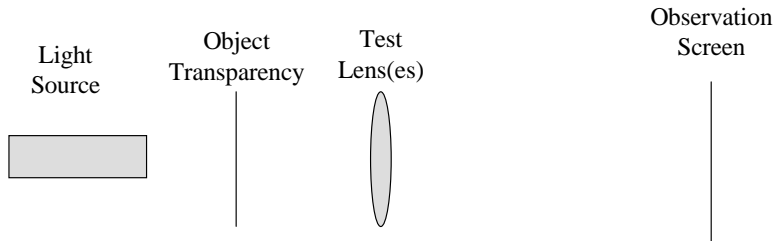


Figure 3: Schematic of the setup

## 1.4 Procedure

Take care to measure distances as accurately as possible in this laboratory, and try to estimate the typical errors in your distance measures. You can do this by taking multiple measurements for a couple of your data points and using as your distance uncertainty  $\frac{\sigma}{\sqrt{N}}$ , as usual. Do not be fooled by how short the procedure section is! Challenge yourself to obtain excellent data that clearly demonstrate the differences between thin and thick lenses. For each of the four lens “systems,” complete the following:

1. Carefully determine the location of the focal points ( $F_1$  and  $F_2$ ) and vertices ( $V_1$  and  $V_2$ ) of your lens system. The easiest way to locate the focal points is to use a small light source and to determine the position where the output beam (after going through the lens system) is collimated (*i.e.*, when you move the screen the size of the beam does not change). Keep the test lens in this fixed position for the remaining steps.
2. Measure the focal length with the aid of the Newtonian lens formula. That is, for several object and image positions, measure  $x_o$  and  $x_i$ . Calculate  $f$  from each pair of data points ( $x_o, x_i$ ), then average to get the final result. Include as your uncertainty  $\frac{\sigma}{\sqrt{N}}$ , as usual.
3. Determine the focal length using the two-magnifications procedure above. Do the measurement a couple of times to get a sense of the uncertainty in  $f$  measured this way.
4. From your measurements, sketch a diagram of the system, labeling the vertices, focal points, and principal points, as well as the distances between them.
5. Use OSLO to model the imaging systems constructed from lenses in your lens kit. Determine the equivalent focal lengths of the system (as calculated by OSLO) and find the rear focal point  $F_2$  by placing the object at  $+\infty$ . You can find the “front” focal point  $F_1$  by reversing the system and again placing the object at  $+\infty$ . Use these values and the equivalent focal length to locate the principal points  $H_1$  and  $H_2$ . You can locate these on the graphs in Step 4.

## 1.5 Analysis

Please include the following items in your report.

1. From your measured data, estimate the uncertainty in your determination of the focal lengths and include these results in your lab writeup in the form; for example, the focal length should be listed as something like  $32.5 \pm 1.0$  mm.
2. How well do the focal lengths that were derived in Steps 2 and 3 agree? Comment on possible sources of any discrepancies.
3. For each of the lenses that you studied, what error is generated by assuming the lens is thin? Given the focal lengths that you measured, you should be able to generate expected curves for  $s_i$  as a function of  $s_o$  and  $f$  using the Gaussian lens formula. For each lens, plot this curve and then overplot a second curve assuming the lens has the same focal length but is “thin” and located midway between the two vertices. In this case, you’ll have to increase your  $s_o$  figures by  $\frac{1}{2}$  the lens thickness, in other words, you’ll be plotting:

$$s_i = \frac{(s_o + \frac{d_\ell}{2})f}{(s_o + \frac{d_\ell}{2}) - f} \quad (6)$$

where  $d_\ell$  is the thickness of the lens or lens system. Finally, using your data from the Newtonian lens formula experiment and noting that  $s_o = f + x_o$  and  $s_i = f + x_i$ : overplot your data points with error bars. Do your data match the thick lens expected curve better?

4. Include your sketches from Step 4 of the Procedure section.